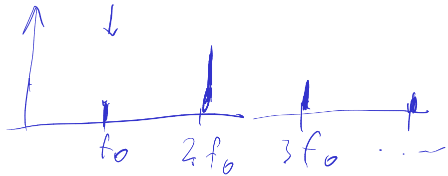
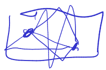
Audio Processing

1. Linear Processing (filters, equalizer, delays effects, modulation)
2. Nonlinear Processing (dynamics processing, distortion, octaver)
3. Time-Frequency Processing
   (a) Phase Vocoder Techniques
   (b) Peak Based Techniques
   (c) Linear Predictive Coding
   (d) Cepstrum



4. Time-Domain Methods
5. Spatial Effects
   (a) Sound Field Methods
   (b) Reverberation
   (c) Convolution Methods



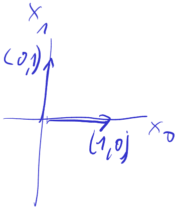
6. Audio Coding

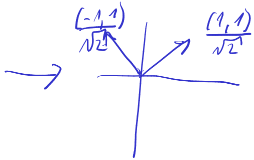
$$\vec{X} = (x_0, x_1)$$

$$\vec{Y} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

$$\vec{Y}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{00} \\ a_{10} \end{pmatrix}$$

$$\vec{Y}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{01} \\ a_{11} \end{pmatrix}$$





$$\boxed{\begin{pmatrix} a_{00} \\ a_{10} \end{pmatrix}^T \begin{pmatrix} a_{01} \\ a_{11} \end{pmatrix} = 0}$$

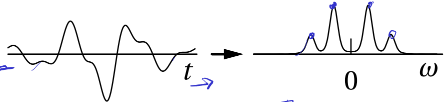
$$\vec{X}\begin{pmatrix} a_{00} \\ a_{10} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underline{x_0} = \begin{pmatrix} a_{00} \\ a_{01} \end{pmatrix}^T \vec{Y}$$

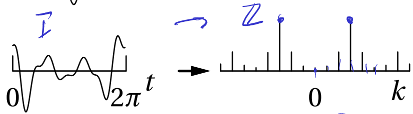
$$x_1 = \begin{pmatrix} a_{10} \\ a_{11} \end{pmatrix}^T \vec{Y}$$

Introduction

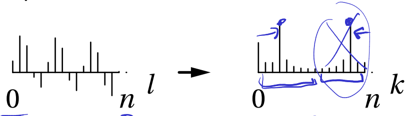
$n = 32$ $F[0]$ $F[1]$ … $F[15]$ $F[16]$

continuous FT


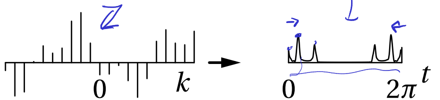
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$$

Fourier series


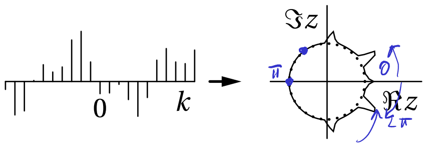
$$F[k] = \int_{0}^{2\pi} f(t)\,\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t$$

DFT (FFT)


$$F[k] = \sum_{l=0}^{n-1} f[l]\,\mathrm{e}^{-\mathrm{i}2\pi kl/n}$$

DTFT


$$F(\omega) = \sum_{k=-\infty}^{\infty} f[k]\,\mathrm{e}^{-\mathrm{i}\omega k}$$

z-transform


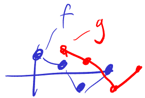
$$F(z) = \sum_{k=-\infty}^{\infty} f[k]\,z^{-k}$$

$z = e^{i\omega}$

Linearity of z-transform:

$$g[t] = af[t] \Rightarrow G(z) = aF(z), \quad g[t] = f_1[t] + f_2[t] \Rightarrow G(z) = F_1(z) + F_2(z).$$

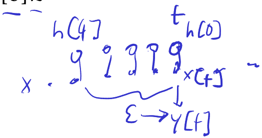
Time delay of 1 $\Rightarrow$ multiplication by z^{-1}:



$$g[t] = f[t-1] \quad \Rightarrow$$

$$G(z) = \sum g[t]z^{-t} = \sum f[t-1]z^{-t} \overset{s=t-1}{=\!=\!=} \sum f[s]z^{-(s+1)}$$

$$= z^{-1}\sum f[s]z^{-s} = z^{-1}F(z)$$



FIR (finite impulse response) filters:

$$y[t] = (h*x)[t] = h[0]x[t] + h[1]x[t-1] + \ldots + h[n]x[t-n] \quad \Rightarrow$$

$$Y(z) = h[0]X(z) + h[1]z^{-1}X(z) + \ldots h[n]z^{-n}X(z)$$

$$= (h[0] + h[1]z^{-1} + \ldots h[n]z^{-n})X(z)$$

$$= H(z)X(z),$$

$h * x \ldots$ convolution of x and h

$H \ldots$ **transfer function**

IIR (infinite impulse response) filters:

$$
\begin{aligned}
y[t] &= (h * x)[t] \\
&= h[0]x[t] + \ldots h[n]x[t-n] \\
&\quad + \hat{h}[1]y[t-1] + \ldots + \hat{h}[m]y[t-m]
\end{aligned}
$$

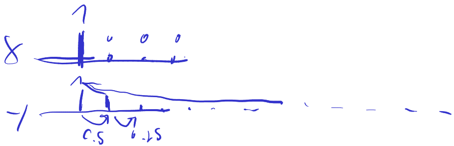
$$y[t] - \hat{h}[1]y[t-1] - \ldots - \hat{h}[m]y[t-m] = h[0]x[t] + \ldots h[n]x[t-n]$$

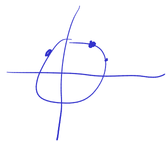
$$(1 - \hat{h}[1]z^{-1} - \ldots - \hat{h}[m]z^{-m})Y(z) = (h[0] + h[1]z^{-1} + \ldots h[n]z^{-n})X(z)$$

$$Y(z) = \frac{h[0] + h[1]z^{-1} + \ldots h[n]z^{-n}}{1 - \hat{h}[1]z^{-1} - \ldots - \hat{h}[m]z^{-m}} X(z)$$

$$Y(z) = H(z)X(z)$$

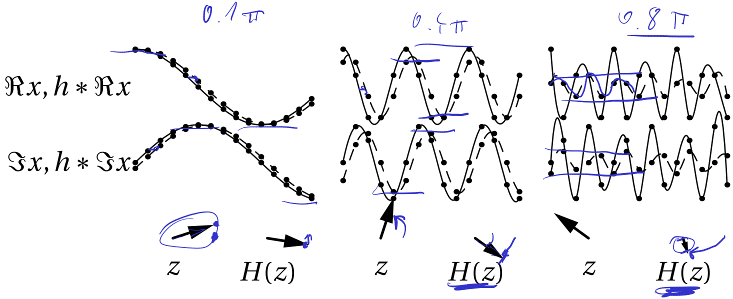
$h = (1)$, $\hat{h} = (0.5)$





(Complex) signal $x = e^{i\omega t} = z^t$, $\omega \in \{0.1\pi, 0.4\pi, 0.8\pi\}$ (solid line)

Filtering (dashed line) by the filter $h = (0.5, 0.5)$ $(H(z) = 0.5 + 0.5z^{-1})$



$z = e^{i0} = 1$

Assume: sampling rate 1 $\Rightarrow$ f from 0 to 0.5 (the Nyquist frequency), $\omega = 2\pi f$ from 0 to π

1 Linear Processing

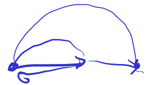
control flow	**signal flow**
controls signal flow	controls signal
slow (every 16 to 4096 samples)	fast

Parametric filters (easy to change properties)

Parametric allpass filter (first order):

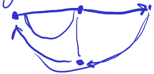
$h[0]$ $h[1]=1$ $h[1]$

$$y[t] = (a * x)[t] = cx[t] + x[t-1] - cy[t-1]$$

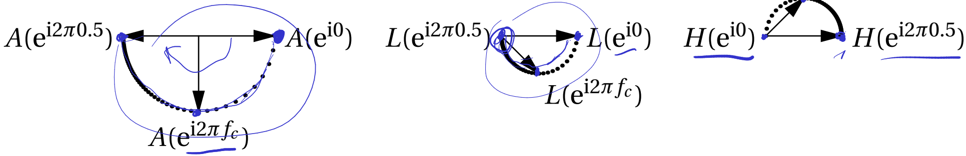


Transfer function:

$$A(z) = \frac{c + z^{-1}}{1 + cz^{-1}}$$



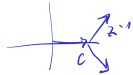
0



Transfer functions for allpass (A), lowpass (L) and highpass (H). $f_c = 0.1$

Magnitude response = 1:

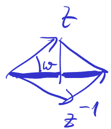
$$|A(z)| = \frac{|c + z^{-1}|}{|1 + cz^{-1}|} = \frac{|c + z^{-1}|}{|z^{-1}| \cdot |z + c|} \overset{|z|=1}{=\!=} 1$$



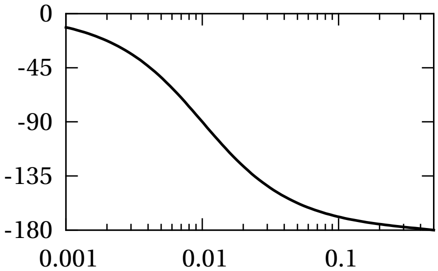
Phase response

$$\varphi = \arg(A(e^{i\omega})) = \begin{cases} 0 & \omega = 0 \\ -90^\circ & \text{“cutoff”-frequency } \omega = 2\pi f_c,\ A(z) = A(e^{-i\omega}) = -i \\ -180^\circ & \text{Nyquist rate } \omega = \pi \end{cases}$$

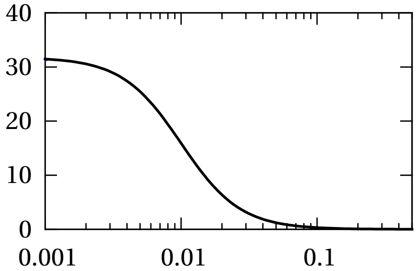
$$\begin{aligned} \frac{c + z^{-1}}{1 + cz^{-1}} &= -i \\ c + z^{-1} &= -i - icz^{-1} \\ c(1 + iz^{-1}) &= -(i + z^{-1}) \qquad \mid \cdot (1 - iz) \\ c(1 + iz^{-1} - iz + 1) &= -(\cancel{i} + z^{-1} + z - \cancel{i}) \\ c(2 + 2\sin\omega) &= -2\cos\omega \\ c &= -\frac{\cos\omega}{1 + \sin\omega} = \boxed{\frac{\tan(\pi f_c) - 1}{\tan(\pi f_c) + 1}} \end{aligned}$$



Phase response of parametric allpass filter with $f_c = 0.01$



Phase



Group delay in samples

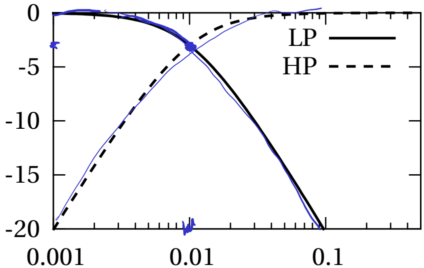
Parametric lowpass:

$$y = l * x = \frac{x + a * x}{2}, \qquad L(z) = \frac{1 + A(z)}{2}$$

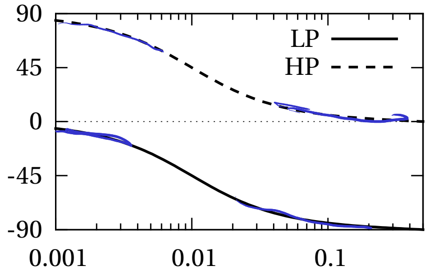
Parametric highpass: substitute − for +, i.e. $h * x = \frac{x - a*x}{2}$

$$L(z) + H(z) = \frac{1 + A(z)}{2} + \frac{1 - A(z)}{2} = \frac{1}{2}\left(2 + \cancel{A(z)} - \cancel{A(z)}\right) = 1$$

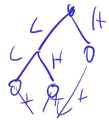
Response of parametric lowpass and highpass filters with $f_c = 0.01$:



Magnitude response in dB



Phase



$dB = 10 \log_{10}$

$-3\,dB \approx \frac{1}{\sqrt{2}}$

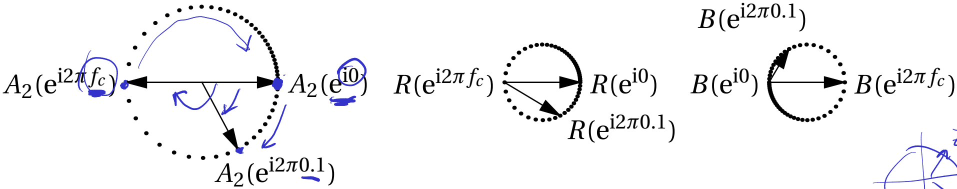
Second-order allpass filter:

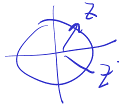
$$y[t] = (a_2 * x)[t] = -d\,x[t] + c(1-d)\,x[t-1] + x[t-2] - c(1-d)\,y[t-1] + d\,y[t-2]$$

Transfer function:

$$A_2(z) = \frac{-d + c(1-d)z^{-1} + z^{-2}}{1 + c(1-d)z^{-1} - d z^{-2}}$$

Transfer functions for second-order allpass (A_2), band-reject (R) and band-pass (B) filters for $f_c = 0.2$ and $f_d = 0.15$:





$z^{-1} = \bar{z}$

Magnitude response $= 1$:

$$|A_2(z)| = \frac{|-d + c(1-d)z^{-1} + z^{-2}|}{|1 + c(1-d)z^{-1} - d z^{-2}|} = \frac{|-d + c(1-d)z^{-1} + z^{-2}|}{|z^{-2}| \cdot |-d + c(1-d)z + z^2|} \overset{|z|=1}{=\!=} 1.$$

$z = a + ib$

$z \cdot \bar{z} = (a+ib)(a-ib)$

$= a^2 + \cancel{iab} - \cancel{iab} + b^2$

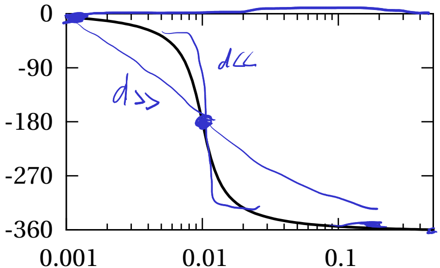
Phase $-180°$ at $\omega = \frac{f_c}{2\pi}$: $A_2(z) = A_2(e^{i\omega}) = -1 \Rightarrow$

$$c = -\cos\omega = -\cos 2\pi f_c$$

Parameter d controls the slope:

$$d = \frac{\tan(\pi f_d) - 1}{\tan(\pi f_d) + 1}$$

Phase response of second-order allpass filter for $f_c = 0.01$ and $f_d = 0.005$:



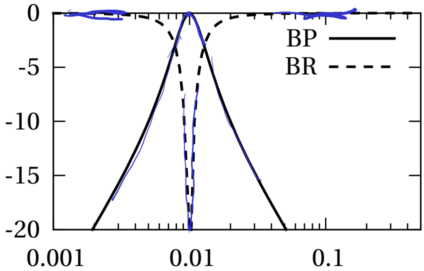
Second-order bandpass filter:

$$y = b * x = \frac{x - a_2 * x}{2}, \qquad B(z) = \frac{1 - A_2(z)}{2}$$

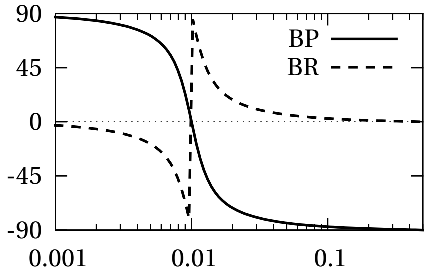
Second-order bandreject filter:

$$y = r * x = \frac{x + a_2 * x}{2}, \qquad R(z) = \frac{1 + A_2(z)}{2}$$

Response of parametric second-order bandpass and bandreject filters with $f_c = 0.01$ and $f_d = 0.005$



Magnitude response in dB



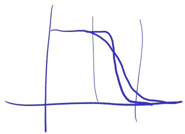
Phase

Second-order lowpass filter ($K = \tan \pi f_c$):

$$y[t] = (l_2 * x)[t] = \frac{1}{1+\sqrt{2}K+K^2}(K^2 x[t] + 2K^2 x[t-1] + K^2 x[t-2]$$
$$- 2(K^2-1)\,y[t-1] - (1-\sqrt{2}K+K^2)\,y[t-2])$$

Second-order highpass filter:

$$y[t] = (h_2 * x)[t] = \frac{1}{1+\sqrt{2}K+K^2}(x[t] - 2x[t-1] + x[t-2]$$
$$- 2(K^2-1)\,y[t-1] - (1-\sqrt{2}K+K^2)\,y[t-2])$$



Shelving filters: add low-/high-pass to original signal.

$$s_l * x = x + \underbrace{(v - 1)}_{-1} l * x\,, \qquad s_h * x = x + (v - 1) h * x\,,$$

v . . . amplitude factor for the passband

Gain in dB $V \quad \Rightarrow v = 10^{V/20}$

$V = 20 \log_{10} v$

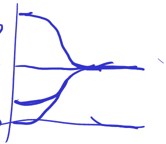
$\frac{V}{20} = \log_{10} v \quad | \, 10^{\circ}$

$10^{\frac{V}{20}} = v$

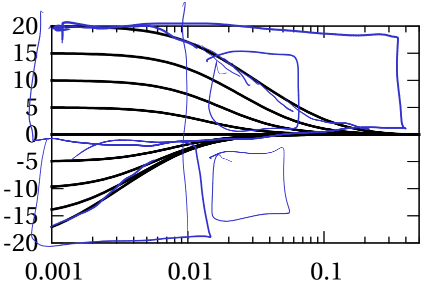
$V = 6$ $\quad v = 2 \rightarrow$

$V = 0$ $\quad v = 1 \rightarrow$

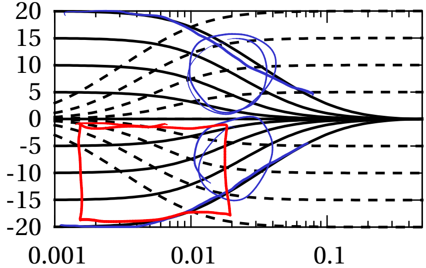
$V = -\infty$ $\quad v = 0 \rightarrow$



Magnitude response of low-frequency and high-frequency shelving filters for gain from −20dB to +20dB and $f_c = 0.01$



uncorrected cut-frequency



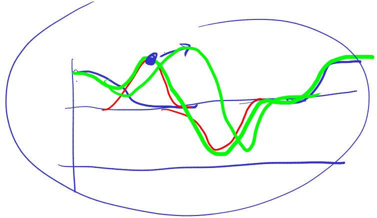
corrected cut-frequency

Correction to make this symmetrical for $v < 1$:

$$c = \frac{\tan(\pi f_c) - v}{\tan(\pi f_c) + v}, \qquad c = \frac{v\tan(\pi f_c) - 1}{v\tan(\pi f_c) + 1}$$

for the low-frequency and the high-frequency filter, respectively.

Peak filter:

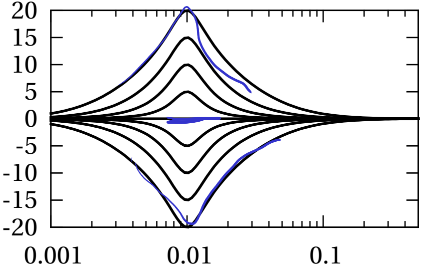


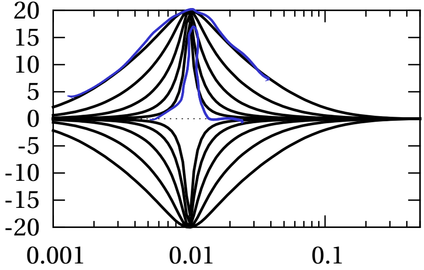
$$p * x = x + (v - 1)\, b * x$$

Similar correction for $v < 1$:

$$d = \frac{\tan(\pi f_d) - \boxed{v}}{\tan(\pi f_d) + \boxed{v}}$$

Magnitude response of peak filters for $f_c = 0.01$:





varying gain, $f_d = 0.005$ varying bandwidth $f_d = 0.0005, 0.001, 0.002, 0.004, 0.008$

Equalizer:

$$e * x \; \cancel{x} = s_l(f_{cl}, V_l) * p(f_{c1}, f_{d1}, V_1) * \cdots * p(f_{cn}, f_{dn}, V_n) * s_h(f_{ch}, V_h) \; \cancel{*} \, x$$

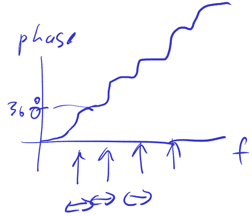
$$e * x$$

$$c = a * b$$

$$\boxed{C = A \cdot B}$$

Phaser: set of second-order bandreject filters with independently varying center frequencies
Implemented by a cascade of second-order allpass filters that are mixed with the original signal

$$ph * x = (1-m)x + m \cdot a_2^{(n)} * \cdots * a_2^{(2)} * a_2^{(1)} * x$$

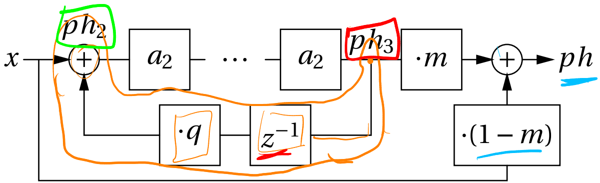


Extension: feedback loop

$$\boxed{ph_3 * x} = a_2^{(n)} * \cdots * a_2^{(2)} * a_2^{(1)} * \boxed{ph_2 * x},$$

$$\boxed{(ph_2 * x)}[t] = x[t] + q \cdot \boxed{(ph_3 * x)}[t-1],$$

$$ph * x = (1-m)x + m \cdot ph_3 * x .$$



Wah-Wah effect: set of peak filters with varying center frequencies

Implemented with a single peak filter with m-tap delay ($W(z) = P(z^m)$)

$p * x = p \cdot x + q \cdot a_2 * x$

$\ldots x(t) \ldots x[t-1] \ldots x[t-2] \ldots a[t-1] \ldots a[t-2]$

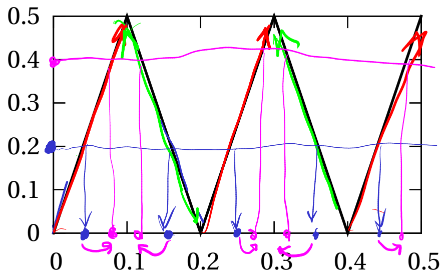
$X \ldots X \cdot z^{-3} \ldots X \cdot z^{-6} \ldots A \cdot z^{-3} \ldots A \cdot z^{-6} = P(z^3)$

Because

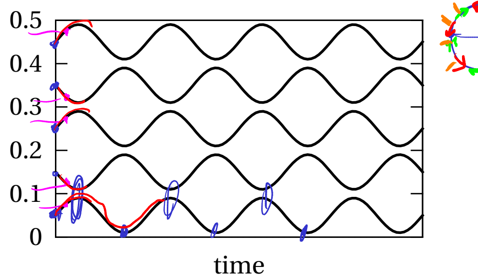
$$|H(e^{i(-\omega)})| = |H(\bar{z})| = |\overline{H(z)}| = |H(z)| = |H(e^{i\omega})|,$$

and because $e^{i\omega} = e^{i(\omega \pm 2\pi)}$ $\Rightarrow$ map $m\omega$ to $[0, \pi]$

$\Rightarrow$ Frequency mapping $f \mapsto g(f)$ so that $|P(e^{i2\pi m f})| = |P(e^{i2\pi g(f)})|$.

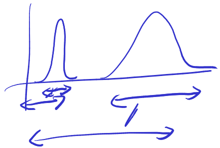


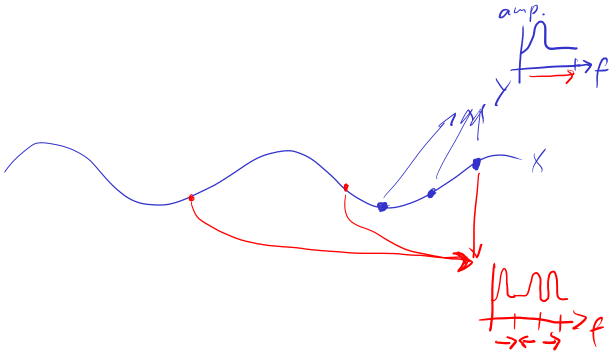
$f \mapsto g(f)$, $m = 5$



Peak frequencies, $m = 5$, controlled by LFO

Constant Q-factor: $q = \frac{f_d}{f_c} \quad \Rightarrow \quad f_d = q\, f_c$

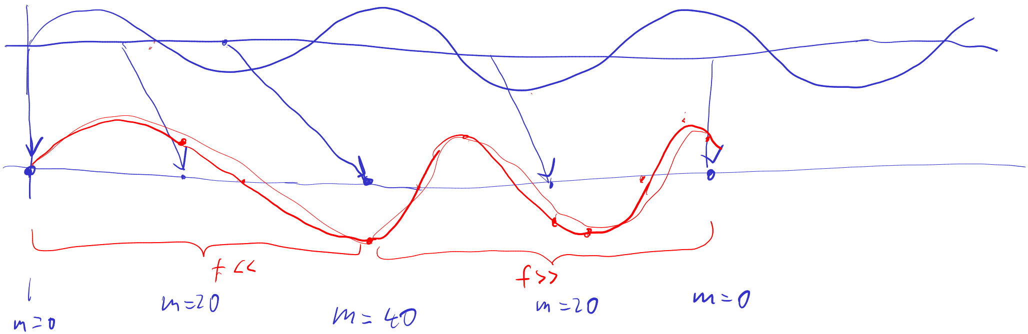




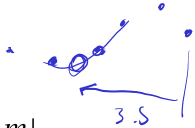
Delay effects: m-tap delay, optional mix with direct signal, optional (IIR-)feedback

$$x[t-1] \rightarrow x[t-m]$$

Example:
vibrato effect: time-shift m varied according to a low-frequency oscillator (LFO) between 0 and 3 ms

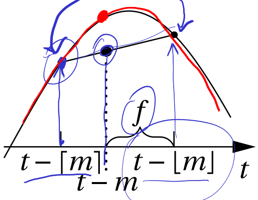


Integer m not fine grained enough $\Rightarrow$ **fractional delays**



1. **linear interpolation**

$$y[t] = (1-f)\,x[t-\lfloor m\rfloor] + f\,x[t-\lceil m\rceil] \qquad f = m - \lfloor m\rfloor$$



t

m

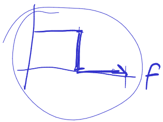
$m = 3.8$

$\lfloor m\rfloor = 3$

$\lceil m\rceil = 4$

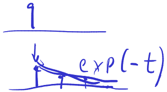
$f = 0.8$

2. correct way: **sinc interpolation**



$$x(s) = \sum_{t=-\infty}^{\infty} x[t]\,\mathrm{sinc}(s-t), \qquad \mathrm{sinc}(s) = \frac{\sin \pi s}{\pi s}$$

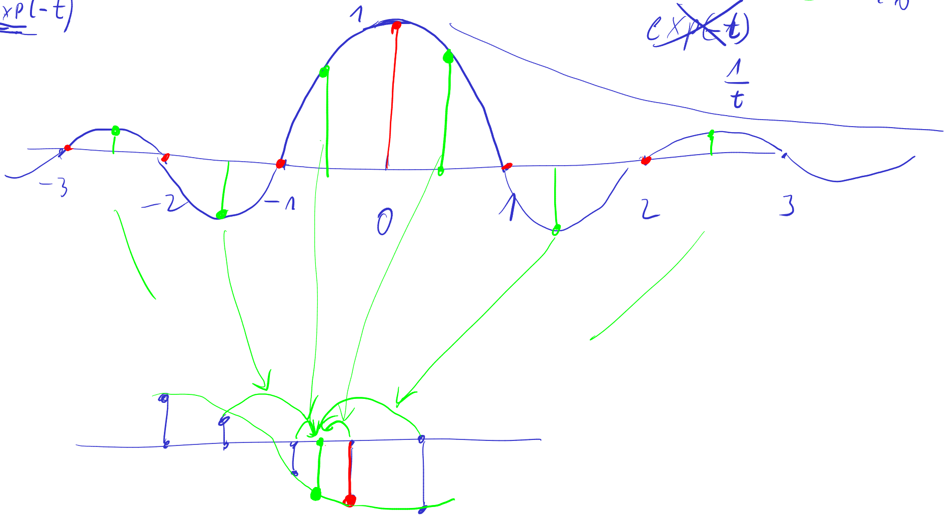
$$y[t] = x[t] + \frac{1}{2} y[t-1]$$



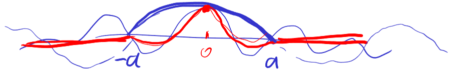
$$\frac{\sin \pi s}{\pi s} \begin{matrix} \to 0 \\ \to 0 \end{matrix}$$

$\cancel{\exp(-t)}$

$$\frac{1}{t}$$



3. finite approximation: **Lanczos kernel**

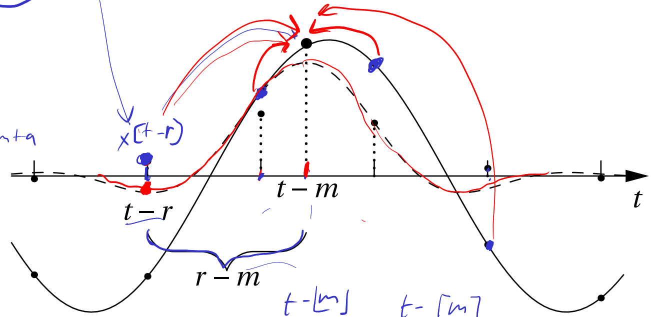


$$y[t] = \sum_{|r-m|<a} x[t-r]\,L(r-m) \qquad L(s) = \begin{cases} \mathrm{sinc}(s)\,\mathrm{sinc}(\frac{s}{a}) & -a < x < a \\ 0 & \text{else} \end{cases}$$

$x(t-m)$

$t-m-a \quad t-r \quad t-m+a$

$|r-m|<a$

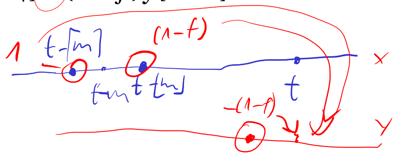


4. **allpass interpolation**

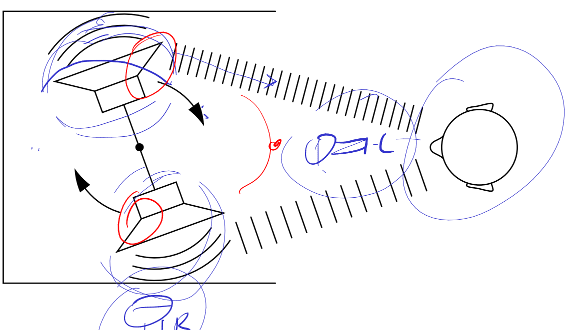
$y(t) = c\,x(t) + x(t-1) - c\,y[t-1]$

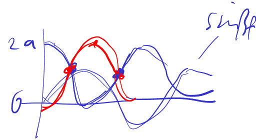
$$y[t] = (1-f)\,x[t-\lfloor m\rfloor] + x[t-\lceil m\rceil] - (1-f)\,y[t-1]$$

5. spline interpolation



Rotary speaker





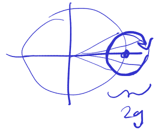
$$y[t] = l(1+\sin\beta t)\,x[t-a(1-\sin\beta t)] + r(1-\sin\beta t)\,x[t-a(1+\sin\beta t)]$$

β ... rotation speed of the speakers
a ... depth of the pitch modulation
l, r ... amplitudes of the two speakers
Stereo effect: l and r unequal but symmetrical values for the left and right channel
e.g. y_l with $l = 0.7, r = 0.5$, y_r with $l = 0.5, r = 0.7$.

Comb filter: delayed signal mixed with direct signal

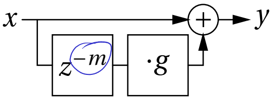
FIR comb filter:

$$y[t] = (c * x)[t] = x[t] + g\,x[t-m], \qquad C(z) = 1 + g z^{-m},$$

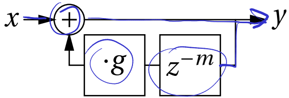


IIR comb filter:

$$y[t] = (c * x)[t] = x[t] + g\,y[t-m], \qquad C(z) = \frac{1}{1 - g z^{-m}},$$



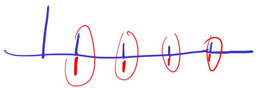
FIR comb filter



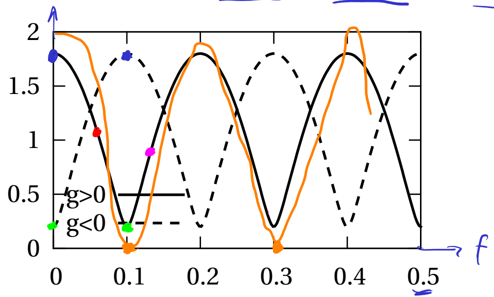
IIR comb filter

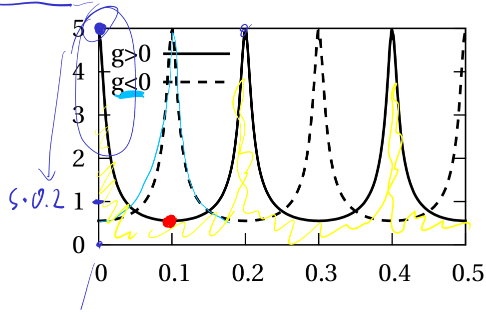
$$(1 + \cancel{g} z^{-m})\left(\frac{1}{1 + \cancel{g} z^{-m}}\right) = 1$$

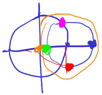
$-g$



Magnitude response with $m = 5$ for $g = 0.8$ and $g = -0.8$:







FIR comb filter

IIR comb filter

$y = x + 0.8y$

$y(1-0.8) = x$

$y = \frac{x}{1-0.8} = \frac{x}{0.2} = 5x$

$y = x - 0.8y$

$y(1+0.8) = x$

$y = \frac{x}{1+0.8} = \frac{x}{1.8}$

Problem: very high gain possible for IIR comb filter. Solution:

- retain L^∞-norm (max): multiply output by $1 - |g|$ $1 - 0.8 = 0.2$
- unmodified loudness for broadband signals: retain L^2-norm: multiply by $\sqrt{1-g^2}$.

$\sqrt{1-0.8^2}$

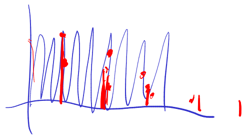
$\sqrt{1-0.64}$

$\sqrt{0.36} = 0.6$

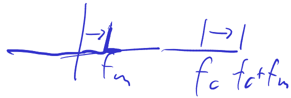
Audio effects with delay filters:

- **slapback** effect: FIR comb filter with a delay of 10 to 25 ms (1950's rock'n'roll)
- **echo**: delays over 50 ms
- **flanger** effect: delays less than 15 ms, varied by a low-frequency oscillator (LFO)
- **chorus** effect: mixing several delayed signals with direct signal, delays independently and randomly varied with LFOs

All effects also possible with IIR comb filters.



Ring modulator: multiplies a carrier signal $c[t]$ and a modulator signal $m[t]$



Complex signals: if $c[t] = e^{i\omega_c t}$ and $m[t] = e^{i\omega_m t}$, then

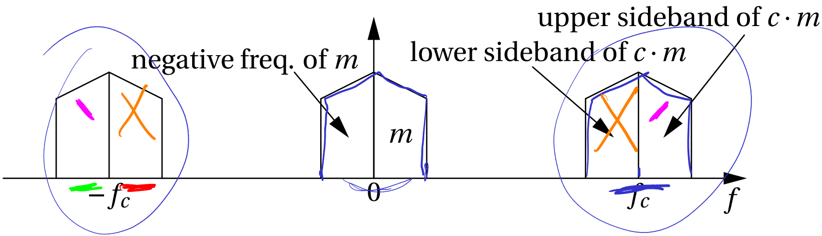
$$c[t]m[t] = e^{i\omega_c t} e^{i\omega_m t} = e^{i(\omega_c+\omega_m)t}$$

Real signals: mirrored negative frequencies included: $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$.

For $c[t] = \cos\omega_c t$ and $m[t] = \cos\omega_m t$:

$$\begin{aligned} c[t]m[t] &= \frac{1}{2}\left(e^{i\omega_c t} + e^{-i\omega_c t}\right)\frac{1}{2}\left(e^{i\omega_m t} + e^{-i\omega_m t}\right) \\ &= \frac{1}{4}\left(e^{i(\omega_c+\omega_m)t} + e^{-i(\omega_c+\omega_m)t} + e^{i(\omega_c-\omega_m)t} + e^{-i(\omega_c-\omega_m)t}\right) \\ &= \frac{1}{2}\left(\cos(\omega_c+\omega_m)t + \cos(\omega_c-\omega_m)t\right). \end{aligned}$$

$e^{ix} = \cos + i\cancel{\sin}$
$e^{-ix} = \cos - i\cancel{\sin}$
$= 2\cos$



Amplitude modulation: reversed roles of c and m $\Rightarrow$ **tremolo** effect

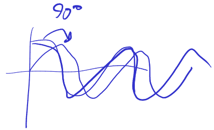
$$y[t] = (1 + \alpha m[t])\, x[t]$$

Getting rid of lower sideband: Reconstruct imaginary part by 90° phase shift filter

90°

$e^{i\omega t} \cdot e^{-i\frac{\pi}{2}}$

$-i$



$\cos\omega t$ should become

$$\cos\left(\omega t - \frac{\pi}{2}\right) = \frac{1}{2}\left(e^{i(\omega t - \frac{\pi}{2})} + e^{-i(\omega t - \frac{\pi}{2})}\right) = \frac{1}{2}\left(-ie^{i\omega t} + ie^{-i\omega t}\right).$$

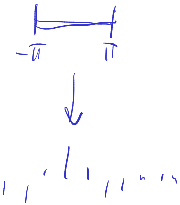
$\Rightarrow$ transfer function of the filter should be

z

$$H(e^{i\omega}) = \begin{cases} -i & \omega > 0 \\ i & \omega < 0. \end{cases}$$

= **Hilbert filter**

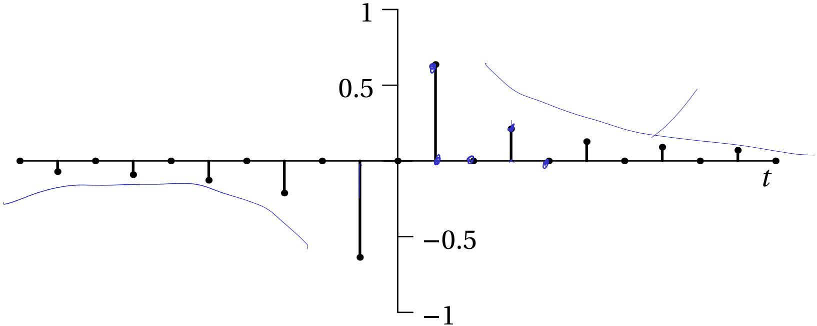
Inverse z-transform $\Rightarrow$ impulse response:



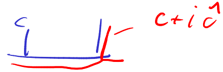
$$h[t] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \boxed{H(\mathrm{e}^{\mathrm{i}\omega})}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi}\left(\int_{-\pi}^{0} \mathrm{i}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega - \int_{0}^{\pi} \mathrm{i}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\right)$$

$$= \frac{1}{2\pi}\left(\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\mathrm{i}t}\bigg|_{-\pi}^{0} - \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\mathrm{i}t}\bigg|_{0}^{\pi}\right) = \frac{1}{2\pi t}\begin{cases} \overbrace{1+1+1+1}^{4} & t \text{ odd} \\ 1-1-1+1 & t \text{ even} \quad = 0 \end{cases}$$

$$= \begin{cases} \dfrac{2}{\pi t} & t \text{ odd} \\ 0 & t \text{ even} \end{cases}$$

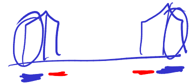


We write $\hat{x} = h * x$.



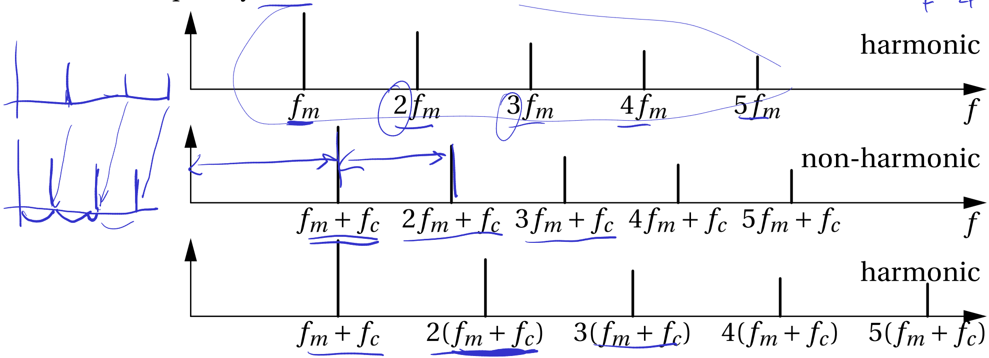
Analytic version (without negative frequencies) of c and m: $c + \mathrm{i}\hat{c}$, $m + \mathrm{i}\hat{m}$. $\Rightarrow$

$$(c + \mathrm{i}\hat{c})(m + \mathrm{i}\hat{m}) = cm - \hat{c}\hat{m} + \mathrm{i}(c\hat{m} + \hat{c}m)$$

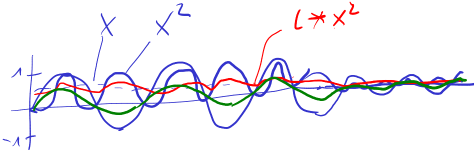


Real part = **single sideband** modulated signal: $cm - \hat{c}\hat{m}$.

Attention: frequency shifts lead to non-harmonic sounds:



2 Nonlinear Processing

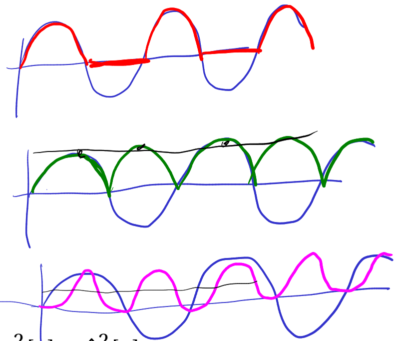


- Linear processing: $y = h * x$
- Nonlinear processing: $y = g(x)$
  - example: $y[t] = (x[t])^2$
  - example: $y[t] = (x[t])^2 + x[t-1] \cdot x[t-2]$
  - example: $y = x(l * x^2)$, low f_c
    $\Rightarrow$ slow amplitude manipulation (dynamics processing)

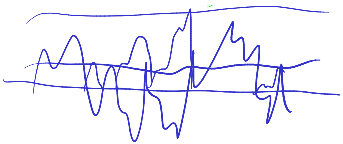
Dynamics processing

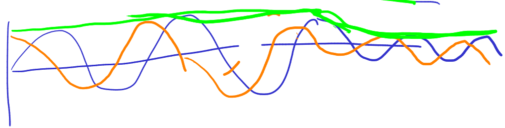
First step: **amplitude follower** comprised of **detector** and **averager**

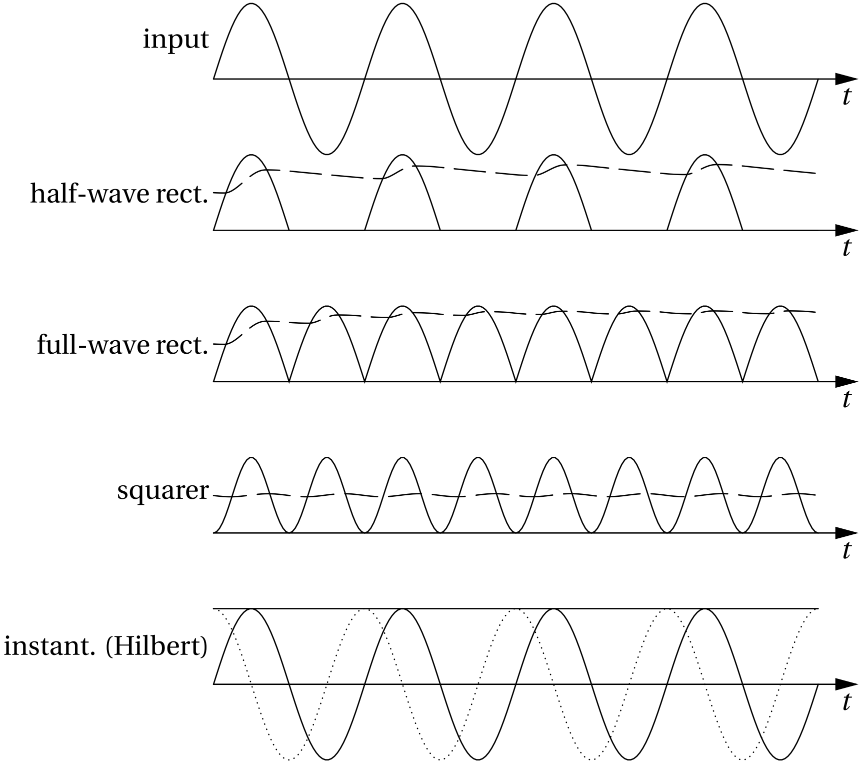
Detector:



- half-wave **rectifier**: $d(x)[t] = \max(0, x[t])$.
- full-wave rectifier: $d(x)[t] = |x[t]|$.
- squarer: $d(x)[t] = x^2[t]$.
- instantaneous envelope (Hilbert transform) $d(x)[t] = x^2[t] + \hat{x}^2[t]$.



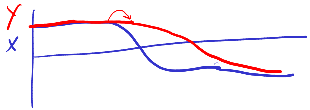



input
half-wave rect.
full-wave rect.
squarer
instant. (Hilbert)
t

Averager:

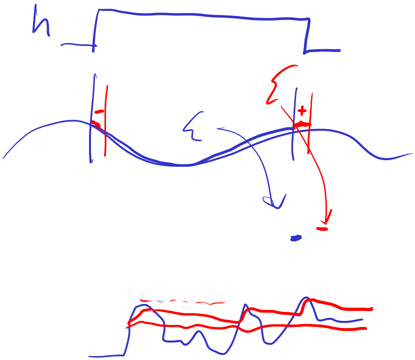
$$y[t] = a(x)[t] = (1-g)\,x[t] + g\,y[t-1], \qquad \text{where} \qquad g = e^{-\frac{1}{\tau}}$$

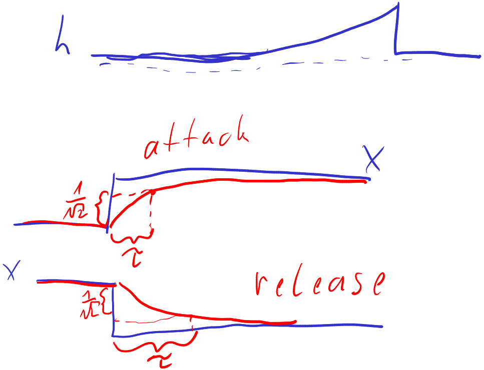
τ … attack and release time constant in samples.



Shorter attack than release times:

$$y[t] = a(x)[t] = \begin{cases} (1-g_a)\,x[t] + g_a\,y[t-1] & y[t-1] < x[t] \\ (1-g_r)\,x[t] + g_r\,y[t-1] & y[t-1] \geq x[t] \end{cases}$$





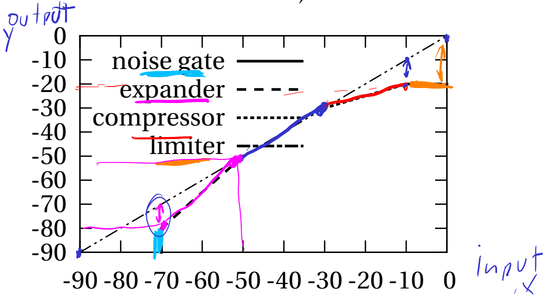
Dynamic range control:

$10^{\left(\frac{d}{20}\right)}$ $r(\ldots)$ $20\log_{10} \to dB$

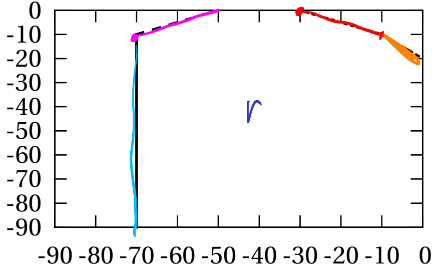
$$y[t] = x[t-\tau]\cdot a_2(\exp(r(\log(a_1(d(x))))))[t]$$

?

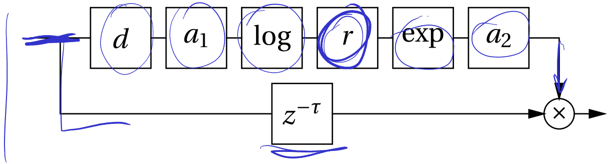
Levels and factors in dB, maximum level is 0 dB:



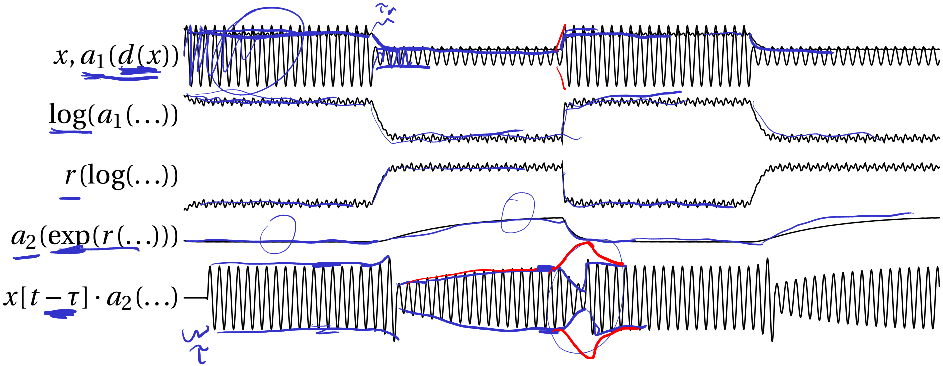
Output level over input level



Gain factor r over input level



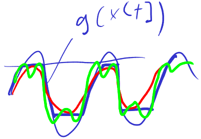
Operator chain for dynamics processing



Compressor/limiter

- **compressor** reduces the amplitude of loud signals
- **expander** does the opposite
- **noise gate** entirely eliminates signals below a threshold
- **limiter** reduces peaks in the audio signal (rectifier as detector)
- **infinite limiter** or **clipper**: limiter with zero attack and release times: $y[t] = g(x[t])$

Typical values: $\tau_{1,a} = 5\,\mathrm{ms}$, $\tau_{1,r} = 130\,\mathrm{ms}$, $\tau_{2,a} = 1\ldots100\,\mathrm{ms}$, $\tau_{2,r} = 20\ldots5000\,\mathrm{ms}$.



$$y[t] = g(x[t])$$

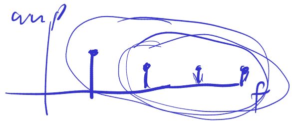
Taylor expansion: $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

Impact on frequency spectrum of a single oscillation:

$$\cos^n(\omega t + \varphi) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos((n-2k)(\omega t + \varphi))$$

$-n \dots th$

$\Rightarrow$ new frequencies $\omega, 2\omega, 3\omega, \dots$ $\pm n\omega$



Total harmonic distortion:

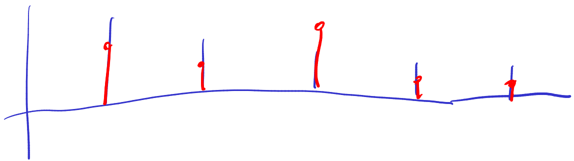
$$\mathrm{THD} = \sqrt{\frac{A_2^2 + A_3^2 + A_4^2 + \dots}{A_1^2 + A_2^2 + A_3^2 + \dots}},$$

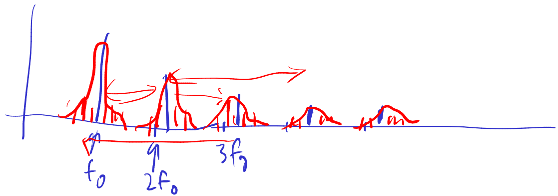
$A_k \dots$ amplitude of frequency $k\omega$

More than one frequency in the input signal:

$$(\cos\omega_1 t + \cos\omega_2 t)^n = \sum_{k=0}^{n} \binom{n}{k} \cos^k \omega_1 t \cos^{n-k} \omega_2 t$$

New frequencies: $a\omega_1 + b\omega_2$ for integers a and b



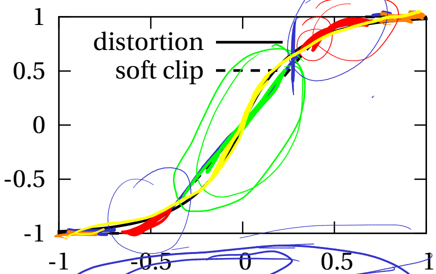


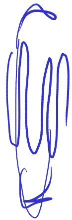
Soft clipping:

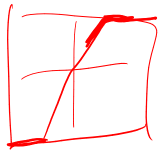
$$g(x) = \mathrm{sign}(x) \cdot \begin{cases} 2|x| & 0 \le |x| \le \frac{1}{3} \\ \frac{3-(2-3|x|)^2}{3} & \frac{1}{3} \le |x| \le \frac{2}{3} \\ 1 & \frac{2}{3} \le |x| \le 1 \,. \end{cases}$$

Distortion:

$$g(x) = \mathrm{sign}(x)(1 - e^{-a|x|}) \qquad a \ldots \text{amount of distortion}$$







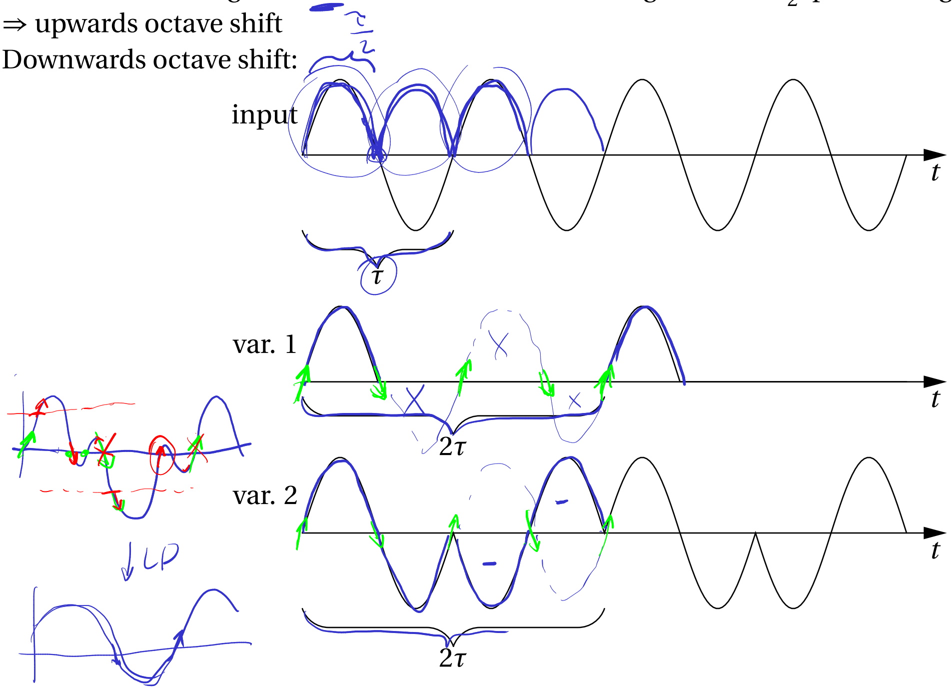
- **overdrive** ... small amount of distortion ("warmer" sound)
- **distortion** ... clearly audible distortion
- **fuzz** ... heavy distortion (mutual interaction between several notes results in noise)
- **exciter** ... light distortion to increase harmonics of a sound (brighter and clearer sound)
- **enhancer** ... like exciter, also uses equalization to shape the harmonic content

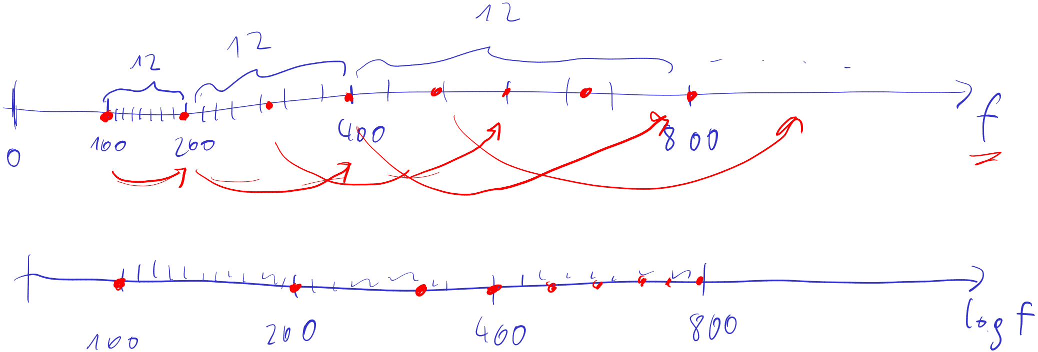
Octaver:

Full-wave rectifier $g(x) = |x|$ sine-wave with wave-length τ into a $\frac{\tau}{2}$-periodic signal

$\Rightarrow$ upwards octave shift

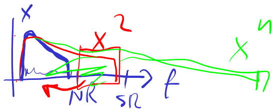
Downwards octave shift:

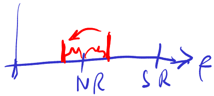


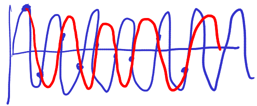


12
12
12
0
100
200
400
800
f
100
200
400
800
log f

Problem: Distortion ⇒ bandwidth extension ($x^n \Rightarrow n\omega$) ⇒ aliasing



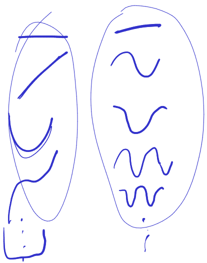


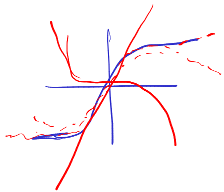


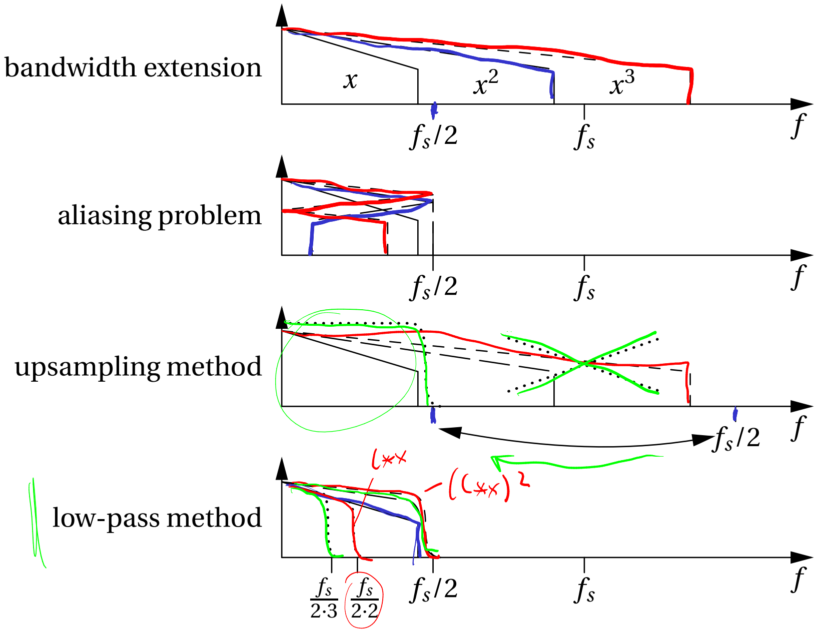
- Solution 1: upsample signal by n using interpolation ⇒ new frequencies from distortion are below the new Nyquist-frequency, afterwards down-sampling (with low-pass filtering)

- Solution 2: split $g(x)$ into $a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, split x into n channels, each low-pass filtered by l_k with a cutoff frequency of $\frac{f_s}{2k}$

$$y[t] = a_1 x + a_2(l_2 * x)^2 + a_3(l_3 * x)^3 + \ldots$$

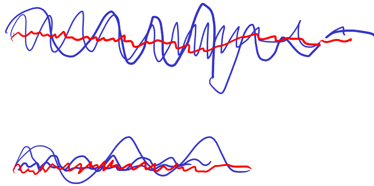






bandwidth extension
x
x^2
x^3
f_s/2
f_s
f
aliasing problem
f_s/2
f_s
f
upsampling method
f_s/2
f
low-pass method
(**
((**)^2
f_s/(2·3)
f_s/(2·2)
f_s/2
f_s
f

3 Time-Frequency Processing

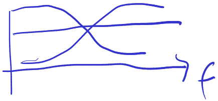


Sinusoidal+residual model:

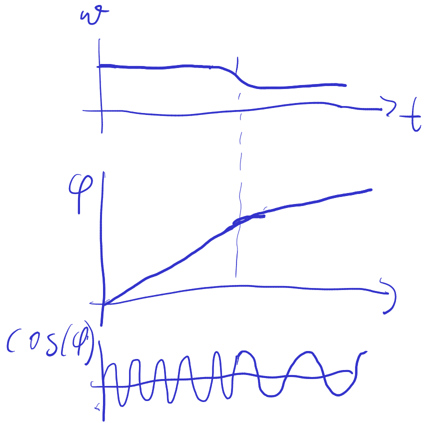
$$x[t] = \sum_k a_k[t]\cos(\varphi_k[t]) + e[t].$$

$a_k[t]$ ... amplitude of the k-th sinusoid
$e[t]$ ... residual signal
$\varphi_k[t]$ ... instantaneous phase of the k-th, which cumulates the instantaneous frequency $\omega_k[t]$:

$$\varphi_k[t] = \sum_{s=0}^{t} \omega_k[s].$$

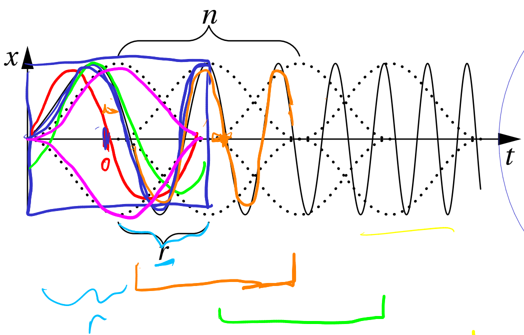


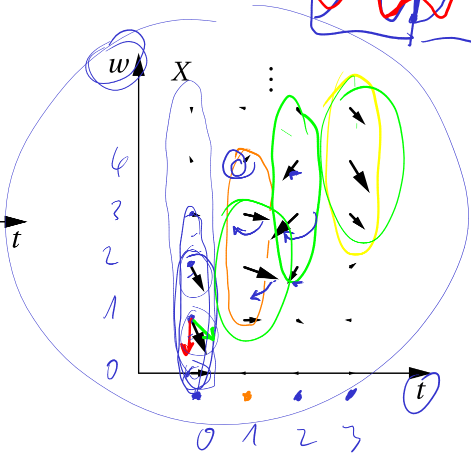
$$\underbrace{\omega t + \varphi}_{\varphi_{abs}}$$

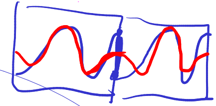


3.1 Phase Vocoder Techniques

Short-time Fourier transform (STFT):





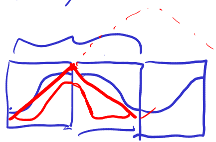


n . . . window or frame size

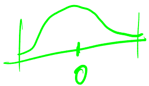
small window ⇔ bad frequency resolution

large window ⇔ bad time resolution and higher latency

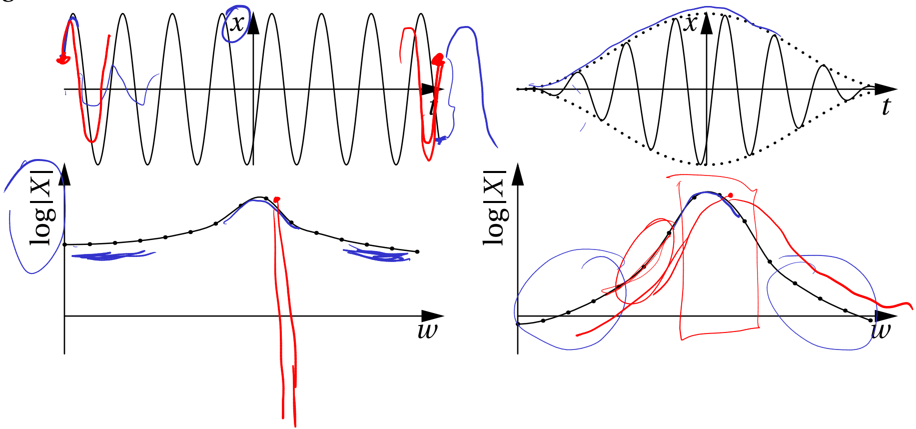
r . . . **hop size** (distance between the centers of consecutive windows)



overlap: $1 - r/n$



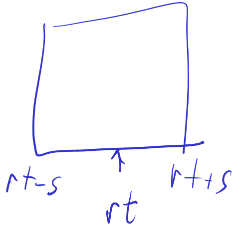
Windowing ($h[t]$):



STFT:

not w, int

$$X[t,w] = \sum_{s=-n/2}^{n/2-1} h[s]\,x[rt+s]\,e^{-i2\pi ws/n}$$



w ... frequency bands/bins (integer, as opposed to ω)
t ... coarser time-resolution ($t+1$ means time shift of r)
$X[t,w] = |X[t,w]|e^{i\varphi[t,w]}$ ... amplitude $|X[t,w]|$, phase $\varphi[t,w]$

Re-synthesis (inverse Fourier-transform, overlap-add method):

$$x[t] = \sum_{s:-\frac{n}{2}\le t-rs<\frac{n}{2}} h_s[t-rs] \sum_w X[s,w]\,e^{i2\pi w(t-rs)}$$

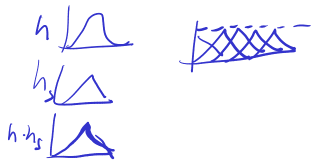
fft
ifft

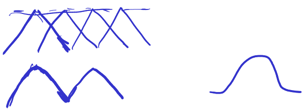
h_s ... synthesis window:

reverses analysis window h

in overlap regions the sum of the resulting windows has to be 1 (summing condition):

$$\sum_s h[t-rs]\,h_s[t-rs] = 1$$

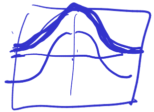




Example: Hann window

$h[t] = \frac{A}{2}(1 + \cos 2\pi t/n)$, hop size $r = n/4$

$h_s = h \quad \Rightarrow \quad \sum_s (h[t - rs])^2 = 1$



$$h^2[t] + h^2[t - n/4] + h^2[t - n/2] + h^2[t - 3n/4]$$

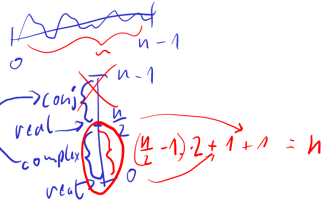
$$= \frac{A^2}{4}(1 + \cos 2\pi t/n)^2 + \frac{A^2}{4}(1 + \cos 2\pi(t/n - 1/2))^2 + \ldots + \frac{A^2}{4}(1 + \cos 2\pi(t/n - 3/4))^2$$

$$= \frac{A^2}{4}(1 + \cos)^2 + \frac{A^2}{4}(1 - \sin)^2 + \frac{A^2}{4}(1 - \cos)^2 + \frac{A^2}{4}(1 + \sin)^2$$

$$= \frac{A^2}{4}(1 + 2\cos + \cos^2 + 1 - 2\sin + \sin^2 + 1 - 2\cos + \cos^2 + 1 + 2\sin + \sin^2)$$

$$= \frac{A^2}{4}(4 + 2(\cos^2 + \sin^2)) = \frac{3A^2}{2} \overset{A=\sqrt{2/3}}{=} 1$$

Phase vocoder = STFT + modifications + inverse STFT

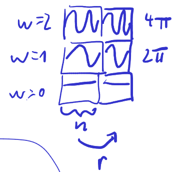


Time stretching: use a different hop size r_s for synthesis
Problem: phases do not match
Solution: **phase unwrapping**:
$\varphi[t,w]$ ... instantaneous phase of $X[t,w]$, so that

r = n

$$X[t,w] = A[t,w]e^{i\varphi[t,w]}$$

If frequency would be exactly w, then the projected phase of $X[t+1,w]$ is



0 ... 2π · w

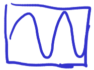
$$\varphi_p[t+1,w] = \varphi[t,w] + 2\pi w r/n \overset{\mathrm{mod}\,2\pi}{=} \varphi[t+1,w]$$

0 ... 2π

Otherwise: unwrapped phase $\varphi_u[t+1,w]$:

$$\varphi_u[t+1,w] = \varphi[t+1,w] \mod 2\pi, \qquad -\pi \le \varphi_u[t+1,w] - \varphi_p[t+1,w] \le \pi$$

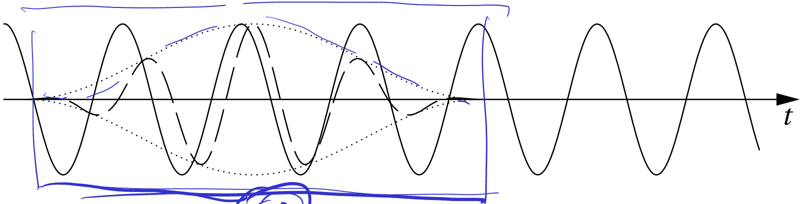
This can be achieved by

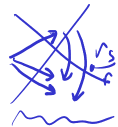


$$\varphi_u[t+1,w] = \varphi[t+1,w] + \mathrm{round}((\varphi_p[t+1,w] - \varphi[t+1,w])/2\pi)\cdot 2\pi$$

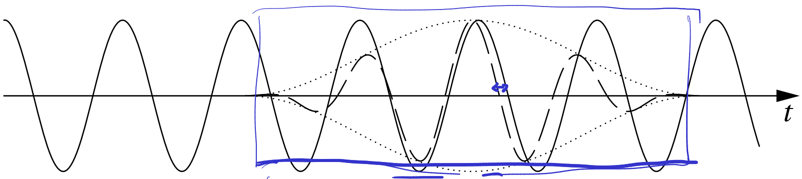
Total phase rotation between t and $t+1$ in frequency bin w:

$$\Delta\varphi[t+1,w] = \varphi_u[t+1,w] - \varphi[t,w]$$

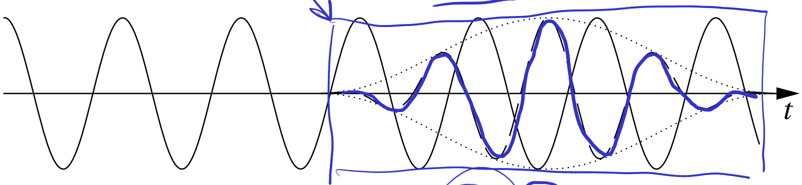




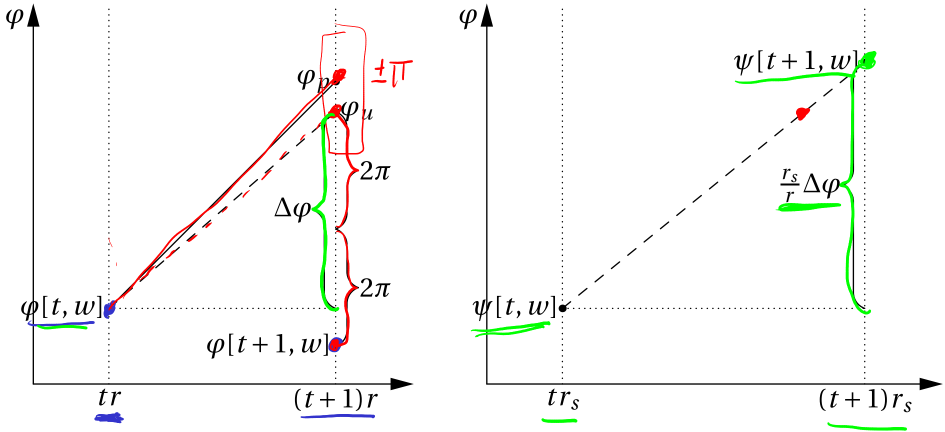
$\varphi[t,w]$



$$\varphi[t+1,w] = \varphi[t,w] + \Delta\varphi \mod 2\pi$$

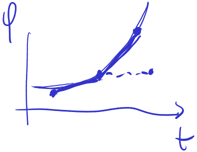


$$\varphi[t,w] + \frac{r_s}{r}\Delta\varphi$$

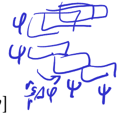


φ
φ_p
$\pm\pi$
φ_u
2π
$\Delta\varphi$
2π
$\varphi[t,w]$
$\varphi[t+1,w]$
tr
$(t+1)r$
φ
$\psi[t+1,w]$
$\frac{r_s}{r}\Delta\varphi$
$\psi[t,w]$
tr_s
$(t+1)r_s$

Time stretching, finally:



$$Y[t,w] = \sum_{s=-n/2}^{n/2-1} h[s]\,y[r_s t + s]\,\mathrm{e}^{-\mathrm{i}2\pi ws/n} = A[t,w]\,\mathrm{e}^{\mathrm{i}\psi[t,w]}$$



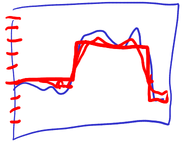
$$\psi[t+1,w] = \psi[t,w] + \frac{r_s}{r}\Delta\varphi[t+1,w]$$

Pitch shifting by time stretching ($r_s = \alpha r$): resampling after time stretching $y[t] = x[\alpha t]$

Problem: Frequency transients and consonants are smeared in time.

Solution: Separate stable from transient components (stable = unchanging phase change):

$$\varphi[t,w] - \varphi[t-1,w] \approx \varphi[t-1,w] - \varphi[t-2,w] \quad \bmod 2\pi$$

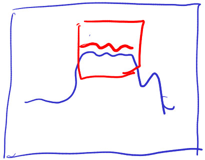


More precisely:

$$|\varphi[t,w] - 2\varphi[t-1,w] + \varphi[t-2,w]| < d \quad \bmod 2\pi$$

where "$|x| < d \mod 2\pi$" means: the smallest $|x + k\cdot 2\pi|$ is smaller than d.

$$|\underbrace{\overset{x}{1.4\pi} - 1\cdot 2\pi}_{-0.1\pi}| = 0.1\pi < d$$



Stable frequency bins: time stretching

Transient bins: drop or use to construct residual signal

Or: do not stretch parts without stable bins

Mutation (**morphing**, **cross-synthesis**, vocoder effect): Use phase of X_1 and magnitude of X_2:

$$Y[t,w] = \frac{X_1[t,w]}{|X_1[t,w]|}|X_2[t,w]|$$

Robotization: Set all phases to zero in each frame and each bin.

Whisperization: randomize the phase

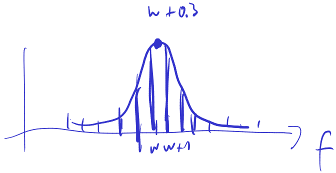
Denoising: attenuate frequency bins with low magnitude, keep high magnitudes unchanged.

$$Y[t,w] = X[t,w]\underbrace{\frac{|X[t,w]|}{|X[t,w]|+c_w}}_{<1}$$

c_w ... controls amount and level of attenuation.

3.2 Peak Based Techniques

- Phase vocoder: represent frequency by frequency bin and phase (bin-number only exact up to f_s/N)
- Peak based: represent frequency by exact peak



Peak detection: fit a parabola to the maximum and the two neighboring bins (in logarithmic representation of the magnitudes)

$a_w = 10\log_{10}|X[t, w_0 + w]|_2^2 \quad (w_0 \ldots \text{bin of local maximum})$

Parabola $p(w) = \alpha w^2 + \beta w + \gamma$ so that $p(w) = a_w$ for $w \in \{-1, 0, 1\}$

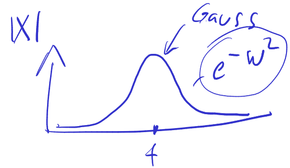
$\Rightarrow \alpha - \beta + \gamma = a_{-1},\ \gamma = a_0,\ \alpha + \beta + \gamma = a_1$

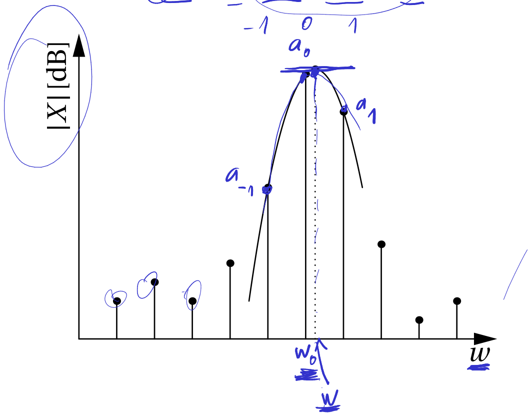
$\Rightarrow \alpha = \frac{1}{2}(a_1 - 2a_0 + a_{-1}),\ \beta = \frac{1}{2}(a_1 - a_{-1})$

Peak of $p(w)$ where $p'(w) = 0 \quad \Rightarrow \quad 2\alpha w + \beta = 0 \quad \Rightarrow$

$$w = -\frac{\beta}{2\alpha} = \frac{a_{-1} - a_1}{2(a_{-1} - 2a_0 + a_1)}.$$

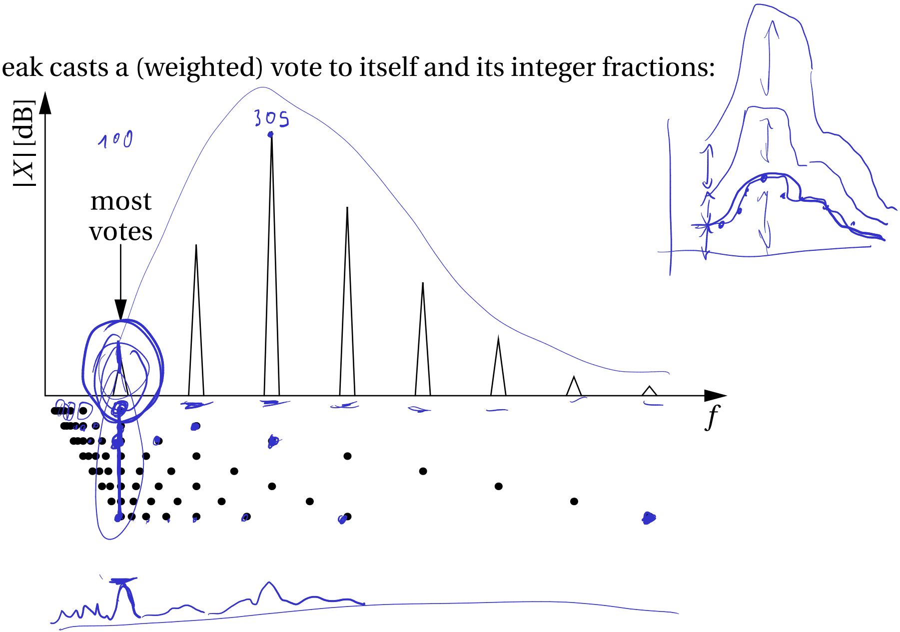
-1 0 1





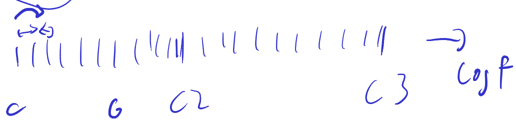
Pitch detection: find the fundamental frequency (integer multiples: harmonics/partials)

Heuristics: Each peak casts a (weighted) vote to itself and its integer fractions:



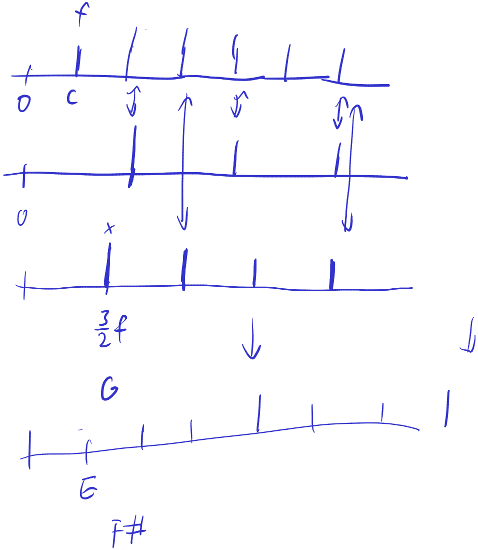
$2^{\frac{1}{12}}$

$\cdot 2^{\frac{1}{12}}$



$$2^{\frac{7}{12}} \approx \frac{3}{2}$$

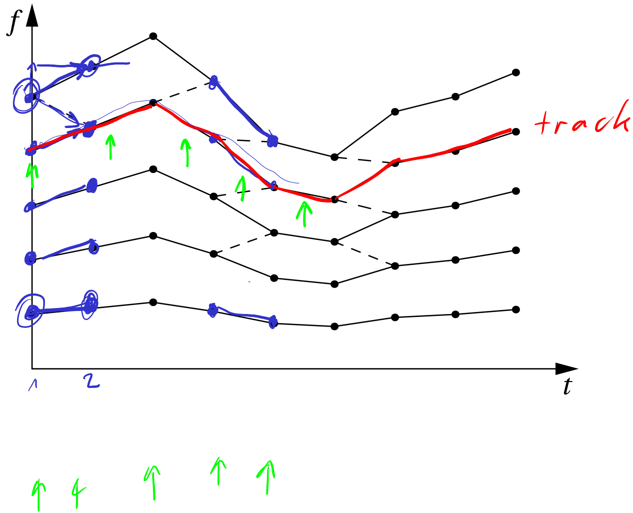
$$1.498 \approx 1.5$$



Peak continuation: associate corresponding peaks of subsequent frames

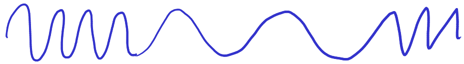
Simple way: choose peak that is closest in frequency (may be wrong in case of transients)

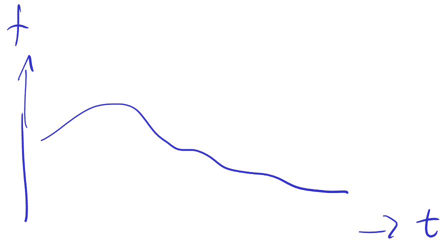
Better way: "guides" – updated to match peaks and fundamental frequency – can be created, killed, turned on/off temporarily

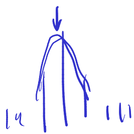


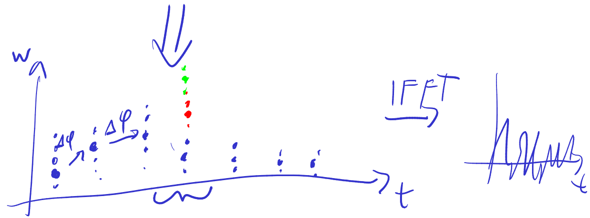
Convert *tracks* representation back to sound (**synthesis**):

- oscillator
- inverse Fourier transform





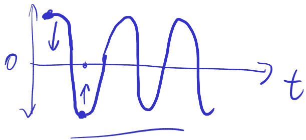




Oscillator (analog, differential equation):

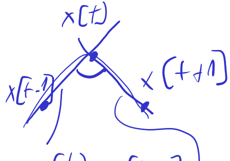
$$\frac{d^2}{dt^2}$$

$$x''(t) = -a x(t)$$



Discretization:

$$x''(t) \approx x[t+1] - 2x[t] + x[t-1]$$



$\Rightarrow$ (**digital resonator**):

$$x[t+1] - 2x[t] + x[t-1] = -a\,x[t]$$

$$x[t+1] = (2-a)x[t] - x[t-1] =: (r * x)[t+1]$$

$$- \quad x[t] - x[t-1]$$

$$+ \quad \underbrace{x[t+1] - x[t]}_{x[t+1] - 2x(t) + x[t-1]}$$

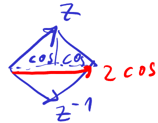
Transfer function:

$$R(z) = \frac{1}{1-(2-a)z^{-1}+z^{-2}} = 0$$

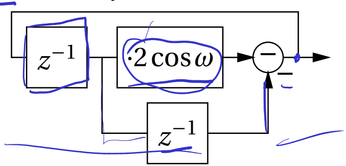
Pole of $R(z)$ is resonance frequency (denominator $= 0$):

$$(2-a)z^{-1} = 1 + z^{-2} \quad | \, z$$

$$(2-a) = z + z^{-1} = 2\cos\omega$$



Initialize by calculating $x[0]$ and $x[1]$ directly



Problem: changes in oscillation energy during frequency changes:

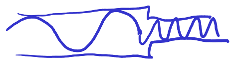
$$E[t] = \overbrace{\underbrace{a x[t]x[t-1]}_{\propto x^2}}^{\text{pot. en.}} + \overbrace{\underbrace{(x[t]-x[t-1])^2}_{v^2}}^{\text{kin. en.}}$$

$$\begin{aligned}
E[t+1] &= ax[t+1]x[t] + (x[t+1]-x[t])^2 \\
&= a((2-a)x[t]-x[t-1])x[t] + ((2-a)x[t]-x[t-1]-x[t])^2 \\
&= a(2-a)x[t]^2 - ax[t]x[t-1] + (x[t]-x[t-1]-ax[t])^2 \\
&= a(2-a)x[t]^2 - ax[t]x[t-1] + (x[t]-x[t-1])^2 - 2ax[t](x[t]-x[t-1]) + a^2x[t]^2 \\
&= a(2-a)x[t]^2 - ax[t]x[t-1] + (x[t]-x[t-1])^2 - a(2-a)x[t]^2 + 2ax[t]x[t-1] \\
&= ax[t]x[t-1] + (x[t]-x[t-1])^2 = E[t]\,.
\end{aligned}$$

Frequency change ($a \mapsto a_2$):

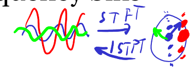
- at signal maximum: $E[t] \approx ax[t]x[t-1] \approx ax[t]^2 \Rightarrow$ changed energy $(\cdot a_2/a)$, same amplitude
- at zero crossing: $E[t] \approx (x[t]-x[t-1])^2 \Rightarrow$ same energy, changed amplitude

This has to be compensated or, better, the signal has to be initialized again.

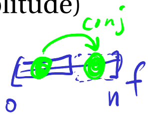


- **Synthesis by inverse Fourier transform**: add spectral pattern of sinusoid to frequency bins

Determine coefficients by forward transform of pure sine wave. Redundancies:



  - amplitudes adjusted by multiplying coefficients (consider only normed amplitude)
  - phase adjusted by multiplication with $e^{i\varphi}$ (consider only normed phase)



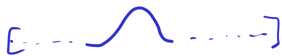
  - all coefficients have same phase (ignore phases)
  - coefficients for two frequencies with an integer bin-distance are the same, just shifted by a certain number of bins (consider only frequencies between bin 0 and 1)
  - coefficients far from the center frequency are negligibly small (consider only small number of bins)

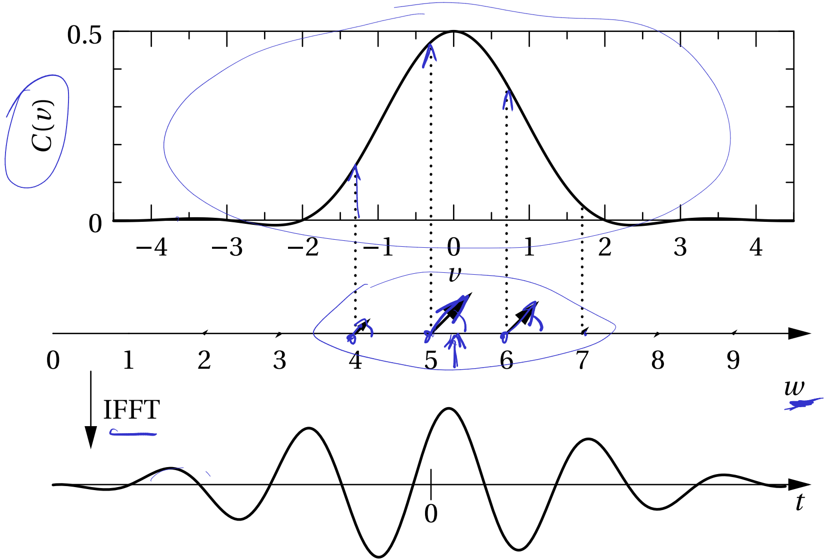
$$C_f[w] = \sum_{s=-n/2}^{n/2-1} h[s] e^{i2\pi f s} e^{-i2\pi w s/n} = \sum_{s=-n/2}^{n/2-1} h[s] e^{-i2\pi (w-nf) s/n},$$

$w = -b, \ldots, b$, $b \ldots$ approximation bandwidth, $nf \in [0,1)$, or better $nf \in [-0.5, 0.5)$

Combine w and f into $v = w - nf$ $\quad\Rightarrow\quad$ zero-padded Fourier transform of window $h[s]$

$$C(v) = \sum_{s=-n/2}^{n/2-1} h[s] e^{-i2\pi v s/n}$$





Spectral motif $C(v)$ for Hann window, used for IFFT synthesis ($nf = 5.3$, $\varphi = \pi/4$)

Copy/add $AC(w - nf)e^{i\varphi}$ into bin w.

Performance comparison:

- one sinusoid:
  - Resonator: $O(1)$ operations per sample
  - inverse FFT: $O(n \log n)$ per frame $\quad\Rightarrow\quad$ $O(\log n)/(1{-}\text{overlap})$ per sample
- k sinusoids:
  - Resonator: $O(k)$
  - inverse FFT: $O(bk/n) + O(\log n)/(1{-}\text{overlap})$

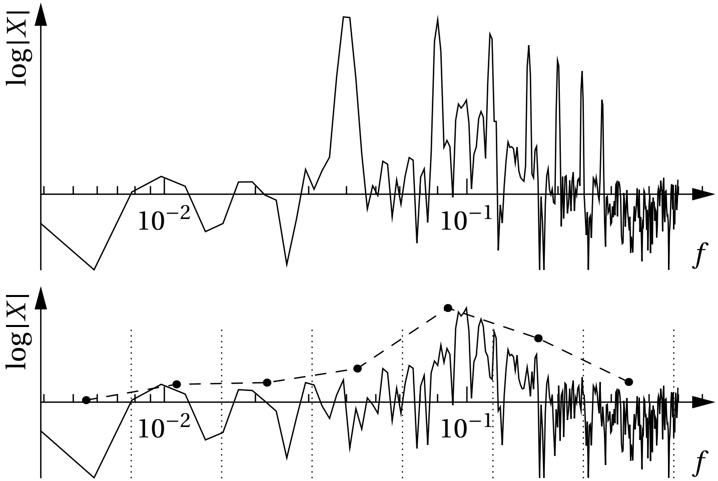
Problem with overlap-add IFFT synthesis: change in frequency $\Rightarrow$ interferences in overlaps
Possible solution: no overlap:

- inverse window $h_s[s] = h[s]^{-1}$
- truncate border (approximation errors mostly near border)
- phases must be exact (avoid phase jumps at border)

Residual signal: subtract re-synthesized signal from the original signal

- in time domain: shorter frames (time resolution more important)
- in frequency domain: no additional FFT needed

Residual signal: stochastic signal (only spectral shape important, no phase information)
Curve fitting on the magnitude spectrum (straight-line segment approximation):

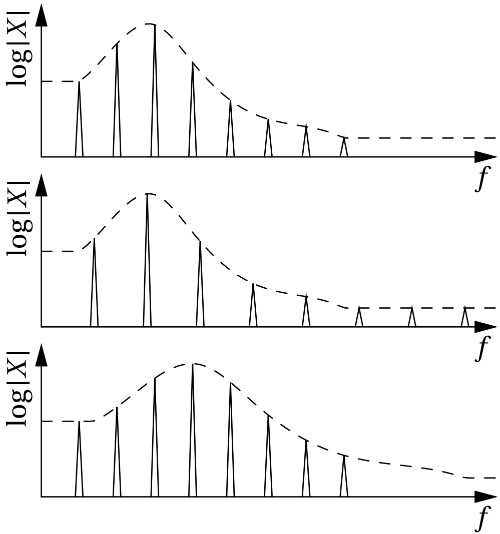


Synthesis of the residual signal:

- convolution of white noise with impulse response of the magnitude spectrum, or
- fill each frequency bin with a complex value: magnitude from the measured magnitude spectrum, random phase.

Applications of peak based methods

- filter with arbitrary resolution
- **Pitch shifting, timbre preservation**



- **Spectral shape shift**

- **Time stretching** (same hop-size but repeat/drop frames)
  avoid smoothing of attack transients: analysis and synthesis frame rates can be set equal for a short time.

- **Pitch correction** (Auto-Tune):

  - detect pitch
  - quantify towards nearest of the 12 semitones
  - sinusoids pitch-scaled by the same factor

- **Gender change**: pitch scaling, move spectral shape along with the pitch for female voice

- **Hoarseness**: increase magnitude of the residual signal

3.3 Linear Predictive Coding

Linear predictive coding (**LPC**):
Prediction filter p: $x[t] \approx (p * x)[t]$
Residual $e[t] = x[t] - (p * x)[t]$

$$(p * x)[t] = p[1]x[t-1] + p[2]x[t-2] + \ldots + p[m]x[t-m]$$

Re-synthesize: $x[t] = (p * x)[t] + e[t]$

If residual $e[t]$ not known exactly ($\tilde{e}[t]$):

$$y[t] = (p * y)[t] + \tilde{e}[t]$$

(all-pole IIR filter)

How to find optimum filter coefficients $p[k]$?
Minimize:

$$E := \sum_t e^2[t] = \sum_t (x[t] - p[1]x[t-1] - p[2]x[t-2] - \ldots - p[m]x[t-m])^2$$

Deriving this with respect to all $p[k]$, setting zero:

$$0 = \frac{\mathrm{d}E}{\mathrm{d}p[k]} = \sum_t 2e[t]\frac{\mathrm{d}e[t]}{\mathrm{d}p[k]} = 2\sum_t e[t]x[t-k] = 2\sum_t \left(x[t] - \sum_j p[j]x[t-j]\right)x[t-k]$$

$$\Leftrightarrow \quad \sum_j p[j]\sum_t x[t-j]x[t-k] = \sum_t x[t]x[t-k]$$

Involves the autocorrelation of x. More stable with windowing:

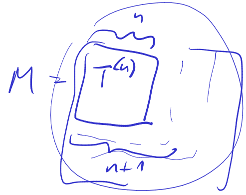
$$r_{xx}[s] := \sum_t w[t]x[t]w[t-s]x[t-s]$$

$\Rightarrow$

$$\sum_j p[j]r_{xx}[k-j] = r_{xx}[k],$$

$\Rightarrow$ equation system with Toeplitz matrix (constant diagonals $M_{k,k-i} = r_{xx}[k-(k-i)] = r_{xx}[i]$)

Levinson-Durbin recursion:
$T^{(n)}$ ... upper left $n \times n$-sub-matrix of $M_{k,j} = r_{xx}[k-j]$
$p^{(n)}$ ... solution vector of $T^{(n)} p^{(n)} = y^{(n)}$ where $y^{(n)} = r_{xx}[1 \dots n]$



$b = \begin{pmatrix} r_{xx}(1) \\ \vdots \\ r_{xx}(n) \end{pmatrix}$

$$T^{(n+1)} \begin{pmatrix} p^{(n)} \\ 0 \end{pmatrix} = \begin{pmatrix} y^{(n)} \\ \epsilon \end{pmatrix} \tag{1}$$

ϵ should be $r_{xx}[n+1]$
Help vector $b^{(n)}$ which satisfies $T^{(n)} b^{(n)} = (0, \dots, 0, 1)$

$$T^{(n+1)} p^{(n+1)} = T^{(n+1)} \underbrace{\left(\begin{pmatrix} p^{(n)} \\ 0 \end{pmatrix} + (r_{xx}[n+1] - \epsilon) b^{(n+1)} \right)}_{p^{(n+1)}} = y^{(n+1)} \tag{2}$$

Find $b^{(n)}$: find also $f^{(n)}$ satisfying $T^{(n)} f^{(n)} = (1, 0, \dots, 0)$

$$T^{(n+1)} \begin{pmatrix} f^{(n)} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \epsilon_f \end{pmatrix}, \qquad T^{(n+1)} \begin{pmatrix} 0 \\ b^{(n)} \end{pmatrix} = \begin{pmatrix} \epsilon_b \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{3}$$

Find α and β so that

$$T^{(n+1)}f^{(n+1)} = T^{(n+1)}\left(\alpha\begin{pmatrix} f^{(n)} \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ b^{(n)} \end{pmatrix}\right) = \alpha\begin{pmatrix} 1 \\ 0 \\ \vdots \\ \epsilon_f \end{pmatrix} + \beta\begin{pmatrix} \epsilon_b \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \tag{4}$$

which can be found by solving

$$\alpha + \beta\epsilon_b = 1, \quad \alpha\epsilon_f + \beta = 0 \qquad \Rightarrow \qquad \alpha = \frac{1}{1-\epsilon_b\epsilon_f}, \quad \beta = -\epsilon_f\alpha \tag{5}$$

Same for $b^{(n+1)}$.
For symmetric Toeplitz matrices: b is just f reversed, and $\epsilon_f = \epsilon_b$.

$\Rightarrow$ Recursion from $n+1 = 1$ to m (length of filter p)
Complexity: $O(m^2)$ (normal equation solving: $O(m^3)$)

Example.

$$x = (1, 2, 1, -1, -2, -1)$$

$$r_{xx}[0] = 1^2 + 2^2 + \ldots,\ r_{xx}[1] = 1 \cdot 2 + 2 \cdot 1 + \ldots, \qquad r_{xx} = (12, 7, -2, -6, -4, -1)$$

To solve for $m = 3$:

$$\begin{pmatrix} 12 & 7 & -2 \\ 7 & 12 & 7 \\ -2 & 7 & 12 \end{pmatrix} \begin{pmatrix} p[1] \\ p[2] \\ p[3] \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ -6 \end{pmatrix}$$

Iteration $n = 0$

$$p^{(1)} = (7/12) = \left(\tfrac{7}{12}\right), \quad f^{(1)} = b^{(1)} = \left(\tfrac{1}{12}\right)$$

Iteration $n = 1$

$$\epsilon_f = \epsilon_b = \frac{1}{12} \cdot 7 = \frac{7}{12} \qquad \Leftarrow (3)$$

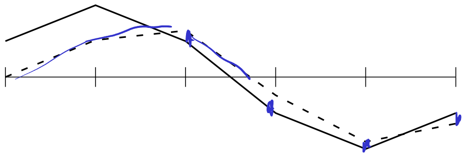
$$\alpha = \frac{1}{1 - \frac{7}{12} \cdot \frac{7}{12}} = \frac{144}{95}, \qquad \beta = -\frac{7}{12} \cdot \frac{144}{95} = -\frac{84}{95} \qquad \Leftarrow (5)$$

$$f^{(2)} = \frac{144}{95} \begin{pmatrix} \frac{1}{12} \\ 0 \end{pmatrix} + \left(-\frac{84}{95}\right) \begin{pmatrix} 0 \\ \frac{1}{12} \end{pmatrix} = \begin{pmatrix} \frac{12}{95} \\ -\frac{7}{95} \end{pmatrix}, \qquad b^{(2)} = \begin{pmatrix} -\frac{7}{95} \\ \frac{12}{95} \end{pmatrix} \qquad \Leftarrow (4)$$

$$\epsilon = \frac{7}{12} \cdot 7 = \frac{49}{12} \qquad \Leftarrow (1)$$

$$p^{(2)} = \begin{pmatrix} \frac{7}{12} \\ 0 \end{pmatrix} + \left(-2 - \frac{49}{12}\right) \begin{pmatrix} -\frac{7}{95} \\ \frac{12}{95} \end{pmatrix} = \begin{pmatrix} \frac{98}{95} \\ -\frac{73}{95} \end{pmatrix} \qquad \Leftarrow (2)$$

$x[t], (p^{(2)} * x)[t]$:



Iteration $n = 2$

$$\epsilon_f = \epsilon_b = \frac{12}{95} \cdot (-2) + \left(-\frac{7}{95}\right) \cdot 7 = -\frac{73}{95} \qquad \Leftarrow (3)$$

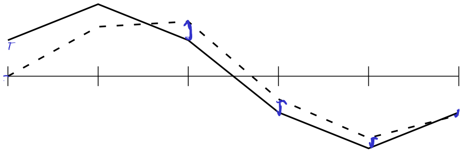
$$\alpha = \frac{1}{1 - (-\frac{73}{95}) \cdot (-\frac{73}{95})} = \frac{9025}{3696}, \qquad \beta = -\left(-\frac{73}{95}\right) \cdot \frac{9025}{3696} = \frac{6935}{3696} \qquad \Leftarrow (5)$$

$$f^{(3)} = \frac{9025}{3696} \begin{pmatrix} \frac{12}{95} \\ -\frac{7}{95} \\ 0 \end{pmatrix} + \frac{6935}{3696} \begin{pmatrix} 0 \\ -\frac{7}{95} \\ \frac{12}{95} \end{pmatrix} = \begin{pmatrix} \frac{95}{308} \\ -\frac{7}{22} \\ \frac{73}{308} \end{pmatrix}, \qquad b^{(3)} = \begin{pmatrix} \frac{73}{308} \\ -\frac{7}{22} \\ \frac{95}{308} \end{pmatrix} \qquad \Leftarrow (4)$$

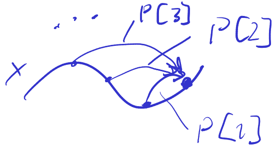
$$\epsilon = \frac{98}{95} \cdot (-2) + \left(-\frac{73}{95}\right) \cdot 7 = \left(-\frac{707}{95}\right) \qquad \Leftarrow (1)$$

$$p^{(3)} = \begin{pmatrix} \frac{98}{95} \\ -\frac{73}{95} \\ 0 \end{pmatrix} + \left(-6 - \left(-\frac{707}{95}\right)\right) \begin{pmatrix} \frac{73}{308} \\ -\frac{7}{22} \\ \frac{95}{308} \end{pmatrix} = \begin{pmatrix} \frac{423}{308} \\ -\frac{27}{22} \\ \frac{137}{308} \end{pmatrix} \qquad \Leftarrow (2)$$

$x[t], (p^{(3)} * x)[t]$:



Two possibilities to apply the predictor p:



- FIR filter $p * x$
- recursive IIR filter $p^{(r)} * x$:

$$y[t] = (p^{(r)} * x)[t] := x[t] + (p * y)[t] = x[t] + p_1 y[t-1] + \ldots + p_m y[t-m]$$

$x \ldots$ "excitation" of $p^{(r)}$

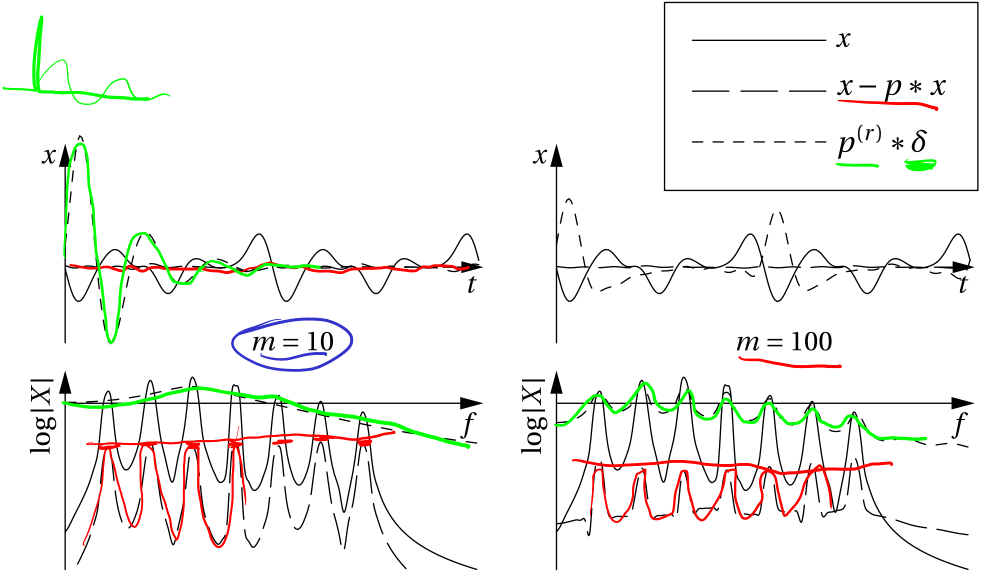
Excited with the prediction residual $\Rightarrow$ original signal is reconstructed:

$$y = p^{(r)} * \underbrace{(x - p * x)}_{\text{res.}} = x - p * x + p * y \Rightarrow y - p * y = x - p * x \Rightarrow y = x$$

x: … 0 0 0 | 0.1 | 0.2 | 0.3 | 0.2 | 0.1 | ....

y: … 0 0 0 | $\underbrace{0 + 0.1}_{0.1}$ | $\underbrace{0.05 + 0.2}_{0.25}$ | $\underbrace{0.125 + 0.01 + 0.3}_{0.426}$ | ....

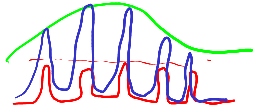
·p[1] ·p[1] ·p[2]



Residual is "whitened" (peaks at same level)
Predictor represents spectral shape ($p^{(r)} * \delta$)

Sound **mutation**:

$$y = \overbrace{p_2^{(r)}}^{\text{voice}} * \overbrace{(x_1 - p_1 * x_1)}^{\text{keyboard}}.$$

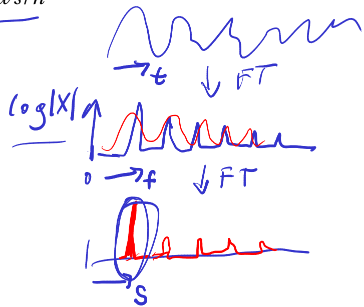


LPC-method widely used in speech analysis, synthesis and compression.

3.4 Cepstrum

Cepstrum (anagram of spectrum): smoothing of the magnitude spectrum by a Fourier method
real cepstrum:

$$\boxed{c[t,s]} := \frac{1}{n} \sum_{w=-n/2}^{n/2-1} \log|X[t,w]| e^{i2\pi ws/n}$$



$s \ldots$ "quefrency"

Low-pass filtering in the s-domain:

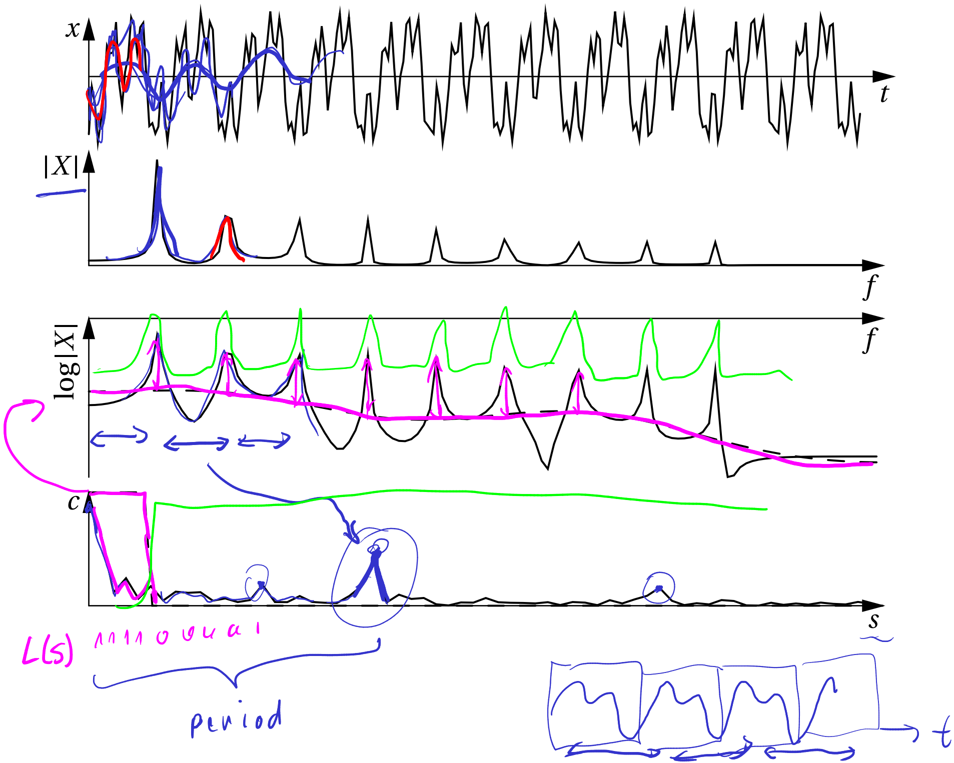
$$l[s] = \begin{cases} 1 & -s_c \le s < s_c \\ 0 & \text{else}, \end{cases}$$

$s_c \ldots$ cutoff quefrency

Forward Fourier transform $\Rightarrow$ smoothed spectrum in the logarithmic domain (dB):

$$\boxed{C_l[t,w]} = \sum_{s=-n/2}^{n/2-1} c[t,s]\, l[s] e^{-i2\pi ws/n}$$

$x \star x = F^{-1}(F(x) \odot \overline{F(x)}) = F^{-1}(|F(x)|^2)$



x
t
|X|
f
log|X|
f
c
s
L(s)
period
t

High-pass window $\boxed{h[s]} = 1 - l[s] \quad \Rightarrow \quad$ complementary source envelope

$$\log|X[t,w]| = C_l[t,w] + C_h[t,w]$$

$$X[t,w] = \exp(C_l[t,w]) \cdot \exp(C_h[t,w]) e^{i\varphi[t,w]}$$

Source-filter separation:

- $\exp(C_l[t,w])$ ... filter or spectral envelope
- $\exp(C_h[t,w])e^{i\varphi[t,w]}$ ... source signal

$C = C_l + C_h$

$e^{C} = e^{C_l} \cdot e^{C_h}$

$e^{C_h} = \frac{e^{C}}{e^{C_l}} = e^{C} e^{-C_l}$

Sound **mutation** (again):

$$\begin{aligned} Y[t,w] &= \exp(C_l^{(1)}[t,w]) \overbrace{\exp(C_h^{(2)}[t,w]) e^{i\varphi^{(2)}[t,w]}}^{X^{(2)} \cdot e^{-C_l}} \\ &= X^{(2)}[t,w] \exp(-C_l^{(2)}[t,w]) \exp(C_l^{(1)}[t,w]) \end{aligned}$$

Formant changing:

$$\begin{aligned} Y[t,w] &= X[t,w] \overbrace{\exp(-C_l[t,w]) \exp(C_l[t,\boxed{w/k}])} \\ &= X[t,w] \exp(C_l[t,w/k] - C_l[t,w]) \end{aligned}$$

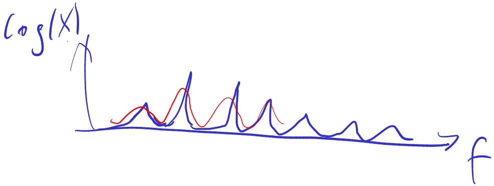
k ... scale factor.

Similar: pitch shifting with timbre preservation

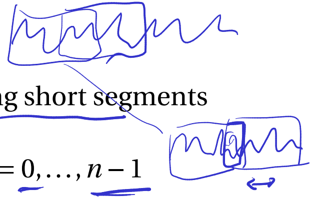
Pitch detection by cepstrum:

Regular intervals of harmonics ⇒ peak at period of fundamental frequency in s-domain

Also peaks for integer multiples ⇒ choose leftmost peak



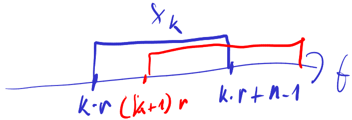
4 Time-Domain Methods



Time stretching in the time domain: shifting overlapping short segments

Overlapping segments:

$$x_k[t] = x[kr + t] \qquad \text{for} \quad t = 0, \ldots, n-1$$



k ... index of the segment

r ... hop-size

n ... segment length

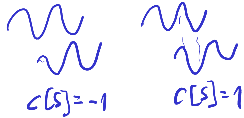
Change hop-size to r' $\Rightarrow$ phase mismatches $\Rightarrow$ amplitude fluctuations

Solution: adjust by additional shift s_k:

$$y[t] = \sum_k x_k[t - kr' - s_k]\, w_k[t - kr' - s_k]$$

w_k ... fade-in/fade-out window

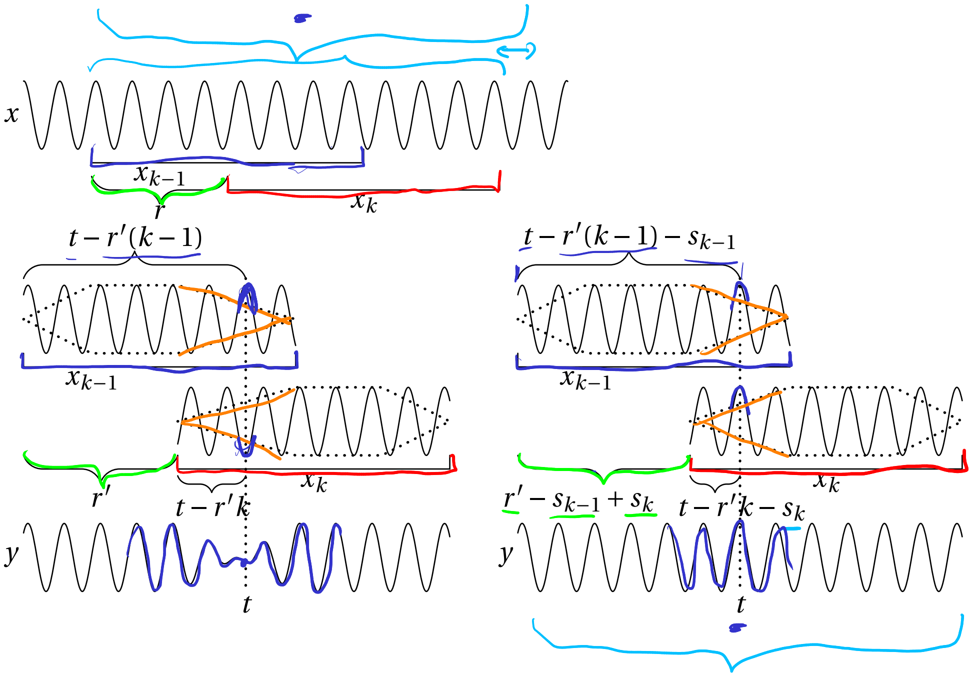
Best fitting shifts s_k by cross-correlation:



$$c[s] = \sum_t x_{k-1}[t + r' - s_{k-1}]\, x_k[t - s] \qquad s_k = \arg\max_s c[s]$$

... **SOLA** (synchronous overlap-add)

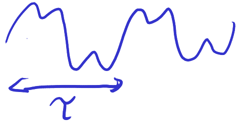
More extreme scaling: repeat/omit segments (source segment $k(l)$ for destination segment l)



x
x_{k-1}
r
x_k
t − r′(k − 1)
x_{k-1}
r′
t − r′k
x_k
y
t
t − r′(k − 1) − s_{k-1}
x_{k-1}
r′ − s_{k-1} + s_k
t − r′k − s_k
x_k
y
t

If pitch is known: **PSOLA** (pitch-synchronous overlap-add)

$r' - r + \cancel{s_k} - s_{k-1}$ must be a multiple of the pitch period τ:



$$s_k = \text{round}\left(\frac{r' - r - s_{k-1}}{\tau}\right)\tau - (r' - r) + s_{k-1}$$

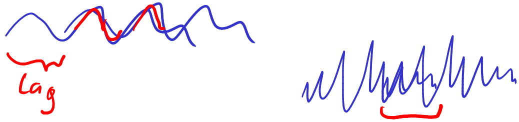
Pitch detection by auto-correlation:

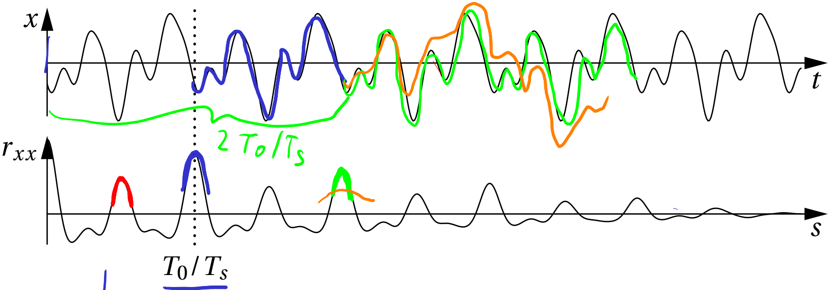
$r_{xx}[s]$: peak at a lag of $s = T_0/T_s$

$T_0 \ldots$ period of the signal ($T_0 = 1/f_0$)

$T_s \ldots$ sampling interval ($T_s = 1/f_s$)

$\Rightarrow \quad s = f_s/f_0$.





partial amplitudes (0.4, 0.8, 0.4, 0.6, 0.1, 0.2, 0.1) $\Rightarrow$ false peak at $0.5 \cdot T_0/T_s$ (strong even partials)

2 4 8

1 2 4

Problems:

– lag is integer $\Rightarrow$ detected fundamental frequencies must not be too high

– fundamental frequency is not the only peak:

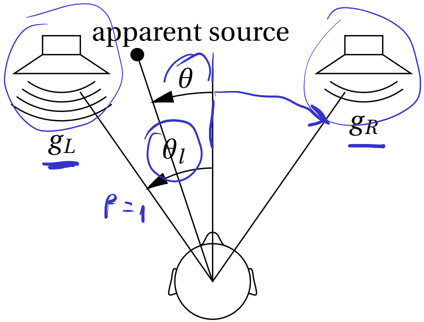
  – integer multiples (T_s-periodic $\Rightarrow$ also kT_s-periodic)

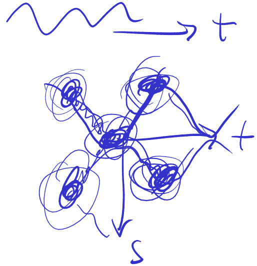
  – integer fractions (harmonics have smaller periods)

5 Spatial Effects

5.1 Sound Field Methods

Panorama:





Apparent source direction

$\vartheta = -\vartheta$

$$p := \frac{\tan\theta}{\tan\theta_l} = \frac{g_L - g_R}{g_L + g_R} = -1$$

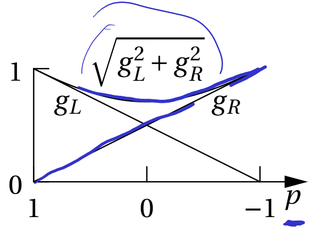
$0 - 1$

$0 + 1$

Linear interpolation (linear panning): “hole” in the center

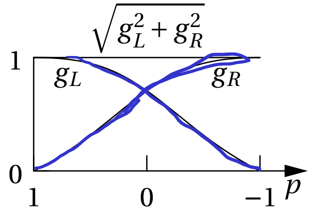
Reason: $\sqrt{E(gx)} = \sqrt{g^2 E(x)} = g\sqrt{E(x)}$, but

$$\sqrt{E(g_L x) + E(g_R x)} = \sqrt{g_L^2 E(x) + g_R^2 E(x)} = \underbrace{\sqrt{g_L^2 + g_R^2}}\sqrt{E(x)}$$



Better:

$$g_L = \frac{1+p}{\sqrt{2(1+p^2)}}, \quad g_R = \frac{1-p}{\sqrt{2(1+p^2)}}$$



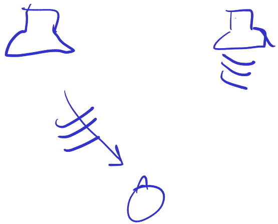
$\Rightarrow$ “overall gain” $\sqrt{g_L^2 + g_R^2} = 1$

$$\sqrt{g_L^2+g_R^2} = \sqrt{\frac{(1+p)^2+(1-p)^2}{\sqrt{2(1+p^2)}^2}} = \sqrt{\frac{1+\cancel{2p}+p^2+1^2-\cancel{2p}+p^2}{2+2p^2}} = \sqrt{\frac{2+2p^2}{2+2p^2}} = 1$$

True for broadband signals and low frequencies

Higher frequencies: different panning

Precedence effect: short delay of up to 1 ms between speakers
⇒ sound appears nearer to speaker that emits sound first
effect strongly depends on the type of sound being played and the frequency

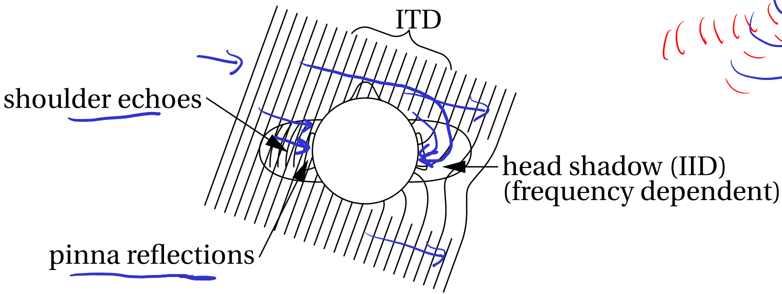


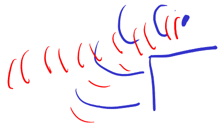
Inter-aural differences (in headphones):

- Inter-aural intensity difference (IID)
  basically a panorama effect
  depends on the frequency (less diffraction of higher frequencies ⇒ more head shadow)

- Inter-aural time difference (ITD)
  time delay between the two channels
  depends on the frequency (below 1 kHz difference is greater, constant otherwise)

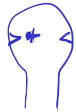
IID and ITD both depend on angle of the sound source



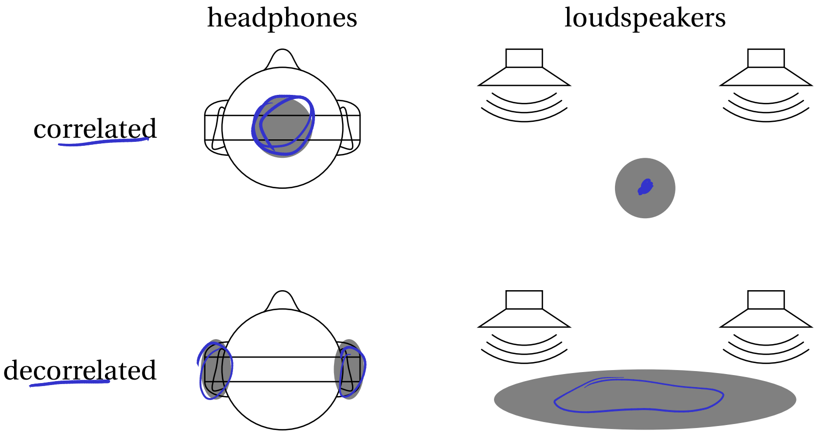


IID + ITD + shoulder echoes + pinna reflections: **head related transfer function** (**HRTF**)
measured by artificial dummy heads at different angles
approximated by IIR filters of an order of about 10

or: approximate head by a sphere:



– calculate the IID as a first-order IIR filter
– ITD implemented by delay
– shoulder echoes by single echo (angle-dependent delay)
– pinna reflections: short series of short-time echoes (very short angle-dependent delays)



Correlation coefficient:

$$r(\tau) = \frac{\int x_L(t)\, x_R(t+\tau)\,\mathrm{d}t}{\sqrt{\int x_L^2(t)\,\mathrm{d}t \int x_R^2(t)\,\mathrm{d}t}}$$

Sound externalization: push apparent sound source out of head

Method: **decorrelation**: complex reverberation or convolution with uncorrelated white noise

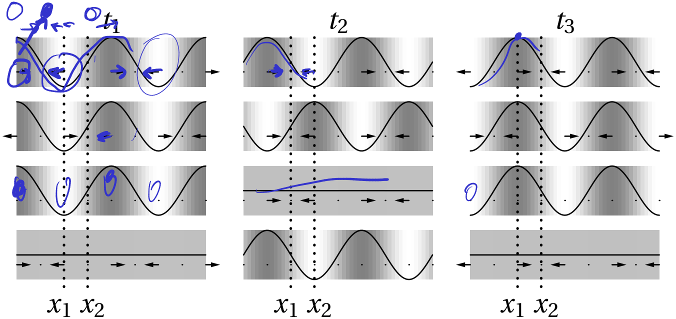
Traveling and standing waves:

right traveling wave

left traveling wave

standing wave (sum)

directional difference



Animation

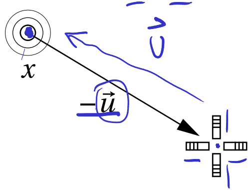
Capture 3D audio: sound field recording

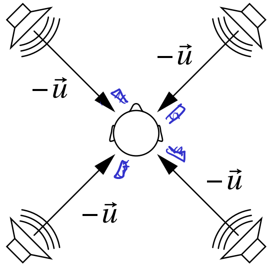
Simple: place microphones and loudspeakers in same directions

Better: **Ambisonics**

– non-directional sound pressure component W

– three directional components X, Y, and Z





$$W = \text{front} + \text{back} + \text{left} + \text{right} + \text{up} + \text{down}$$

$$X = \text{front} - \text{back}$$

$$Y = \text{left} - \text{right}$$

$$Z = \text{up} - \text{down}$$

$$(W, X, Y, Z) = (\sqrt{2}/2, \vec{u}) \cdot x$$

Loudspeaker at direction $\vec{u}$:

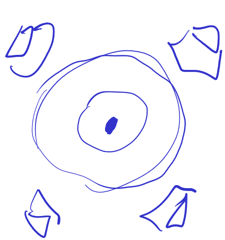
$$\frac{1}{2}(G_1 W + G_2 (X, Y, Z)^\top \vec{u})$$

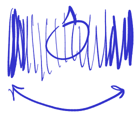
G_1, G_2 depend on the theory (there are several), frequency-dependent (filters)

Disadvantage: “sweet spots”

⇒ Higher-order versions of Ambisonics (higher derivatives) ⇒ wider sweet spots

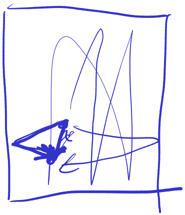
If elevation component is not needed ⇒ ignore Z channel





5.2 Reverberation

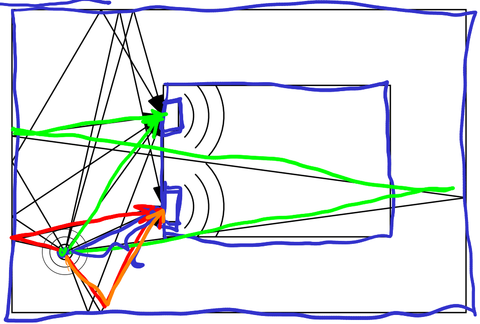
Apparent distance of sound from the listener, room size:
– direct sound
– reflections from walls
– ratio of direct to reverberating sound
  – direct sound loses energy with distance
  – reverberating sound fills room continuously



Direct sound delay T_d, reflection delay T_r $\Rightarrow$ cue for position

Problem: additional reverberation in room of listener

Robust method: **room-within-a-room** model.



$A \propto r^2$

$E \propto \frac{1}{A} \propto \frac{1}{r^2}$

$\text{gain} \propto \sqrt{E} \propto \frac{1}{r}$

– virtual holes in wall at loudspeaker positions

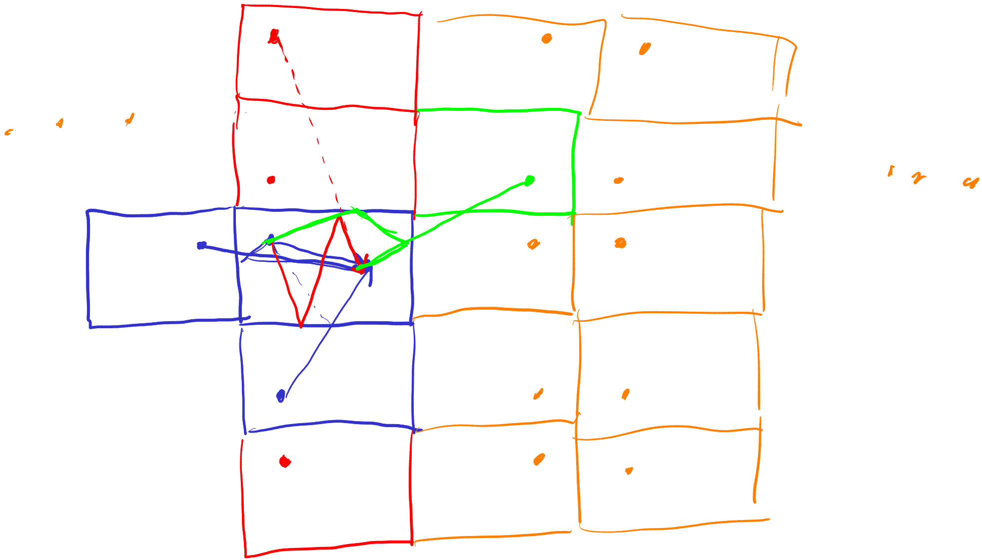
– delay according to the path length l from source to hole (delay = l/c, c . . . speed of sound)

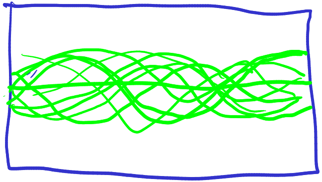
– paths may include reflections of the outer room

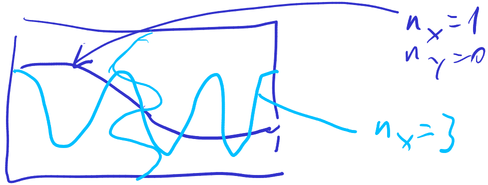
– gain set to $1/l$ (l in meters) (reason: spherical sound waves)

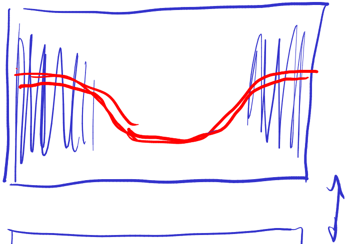
– gain limited to 1 to avoid infinite (or too high) gains

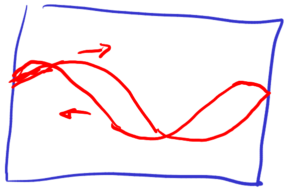
– attenuate if sound direction is opposite to speaker direction

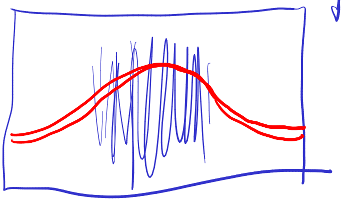





$n_x = 1$
$n_y = 0$
$n_x = 3$







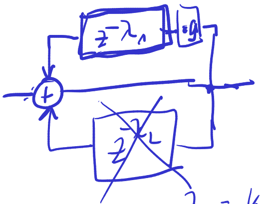
Problem: sound path calculation for multiple reflections computationally demanding
However: sound waves become increasingly planar and aligned with room geometry

Normal modes: standing waves in room

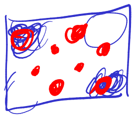
For room of size (l_x, l_y, l_z):
mode number vector (n_x, n_y, n_z) $(n_i = 0, 1, \ldots)$ corresponding to wave length

$$\lambda_n = 2\left(\left(\frac{n_x}{l_x}\right)^2 + \left(\frac{n_y}{l_y}\right)^2 + \left(\frac{n_z}{l_z}\right)^2\right)^{-\frac{1}{2}}$$

Impulse response of room: resonances at frequencies $f_n = c/\lambda_n$
For irreducible triplets n: fundamental frequency + multiples $\Rightarrow$ harmonic frequencies
$\Rightarrow$ implemented by comb filters

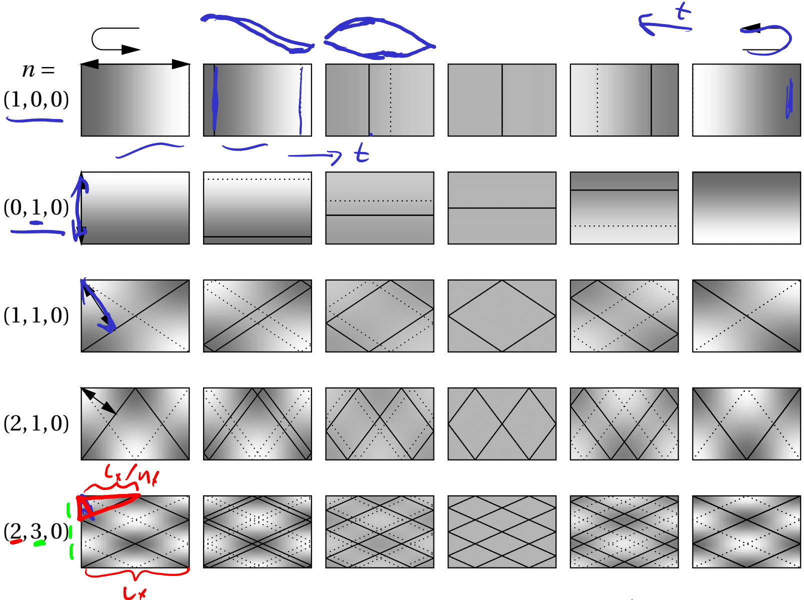


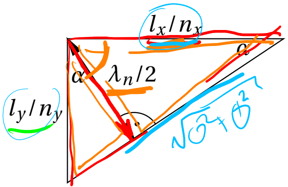
$\lambda_2 = k \cdot \lambda_1$



$n_x = 1$
$n_y = 1$

$\cancel{n_x = 2}$
$\cancel{n_y = 2}$





$$\frac{\lambda_n/2}{l_y/n_y} = \frac{l_x/n_x}{\sqrt{(l_y/n_y)^2+(l_x/n_x)^2}}$$

$$\Rightarrow \quad \lambda_n = 2\frac{1}{\sqrt{(n_x/l_x)^2+(n_y/l_y)^2}}$$

Animations:

(1, 0, 0)

(0, 1, 0)

(1, 1, 0)

(2, 1, 0)

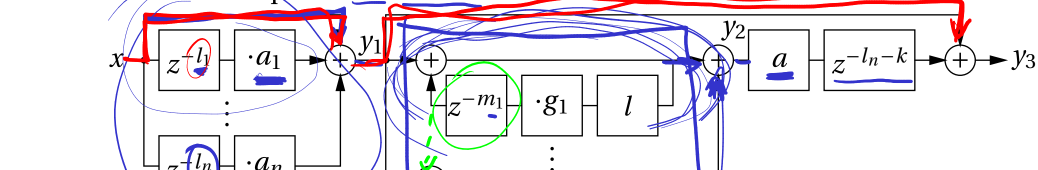
(2, 3, 0)

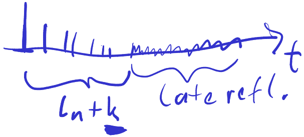
Reverberation without “coloration” (flat magnitude response): delay-based all-pass filter:

$$y[t] = (a * x) = cx[t] + x[t-m] - cy[t-m]$$

$$y[t] = cx[t] + x[t - \underset{m}{\cancel{x}}] - cy[t - \underset{m}{\cancel{x}}]$$

Combination of techniques: **Moorer's reverberator**.





Early reflections (delays l_i based on the sound trajectories):

$$y_1[t] = x[t] + a_1 x[t-l_1] + \ldots + a_n x[t-l_n]$$

IIR comb filters with a low-pass filter in the loop:

$$y[t] = (c * x)[t] = x[t] + g(l * y)[t-m]$$

applied in parallel for late reflections:

$$y_2[t] = c_1 * y_1 + c_2 * y_1 + \ldots + c_o * y_1$$

(m_i are based on wavelengths of room modes, low-pass filter simulates the behavior of the walls)

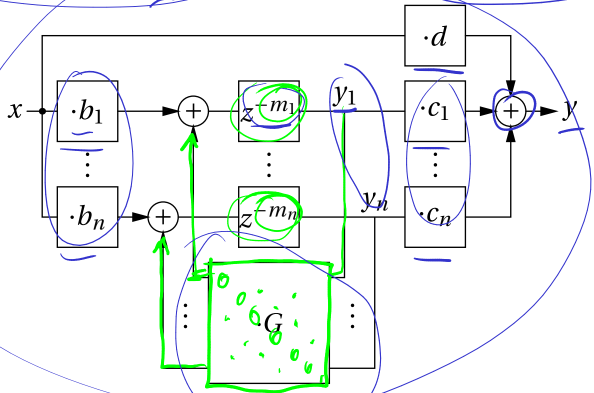
fed into all-pass filter, delayed and mixed together:

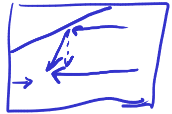
$$y_3[t] = y_1[t] + (a * y_2)[t - l_n - k]$$

Generalization of recursive comb filter $y[t] = x[t] + g \cdot y[t-m]$: **feedback delay network** (**FDN**)
g substituted by a matrix G:

$$\vec{y}[t] = x[t-\vec{m}]\vec{b} + G\vec{y}[t-\vec{m}] \qquad \text{and} \qquad y[t] = d\,x[t] + \vec{c}^{\mathsf{T}}\vec{y}[t]$$

($\vec{y}[t-\vec{m}]$ means: each component of $\vec{y}$ is delayed by a different delay m_i)





If G is a diagonal matrix $\Rightarrow$ set of parallel comb filters as in Moorer's reverberator
Non-diagonal elements of G: interaction between the room's normal modes (due to diffusive elements)

Taking the z-transform:

$$\vec{Y}(z) = \operatorname{diag}\left(z^{-\vec{m}}\right)\left(\vec{b}X(z) + G\vec{Y}(z)\right),$$

$$\left(\operatorname{diag}\left(z^{\vec{m}}\right) - G\right)\vec{Y}(z) = \vec{b}X(z),$$

$$H(z) = \frac{Y(z)}{X(z)} = d + \vec{c}^{\mathsf{T}}\left(\operatorname{diag}\left(z^{\vec{m}}\right) - G\right)^{-1}\vec{b}$$

Poles: $\det(\operatorname{diag}(z^{\vec{m}}) - G) = 0$

– should be inside unit circle to achieve a stable system)

– should have same absolute value (modes will decay at the same rate ⇒ no "coloration")

– first lossless prototype (poles on unit circle, e.g. G unitary matrix)

– attenuation coefficients α^{m_i} in feedback loops

– make higher frequencies decay faster (attenuation coefficients now lowpass filters)

– Feedback matrices of special form (fast implementation, e.g. circular Toeplitz matrices ⇒ Fourier methods)

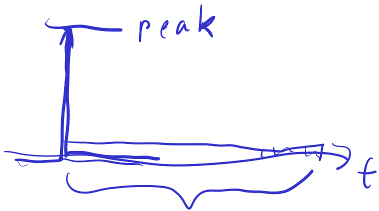
5.3 Convolution Methods

Real room reverberation: convolve the input signal with **room impulse response**

How to determine room impulse response?
Simple: emit impulse (at source position), record result (at listener position)
Problem: large signal peak, little sound energy

Crest factor:

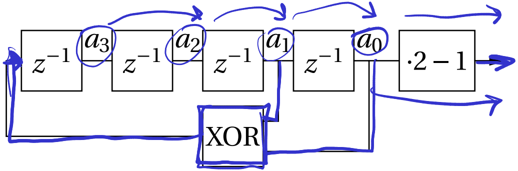
$$C = \frac{\text{peak}|x|}{\text{RMS}(x)}$$



Solution: **maximum length sequences** (**MLS**) (pseudo-random binary (bit) sequences, generated by linear feedback shift registers)

Example (shift register of size 4 (a_3, a_2, a_1, a_0)):

$$a_3[t] = a_0[t-1] \text{ XOR } a_1[t-1], \qquad a_k[t] = a_{k+1}[t-1] \quad \text{for} \quad k = 0, 1, 2\,.$$



For initial values 0001 for a, the result is

$$a_0[t] = 100010011010111\,100010011010111\,100010011010111\ldots$$

a_0 a_1 a_2 a_3

1 0 0 0 1 0 0 1 1 0 1 0 1 1 1 1 0 0 0

1 0 0

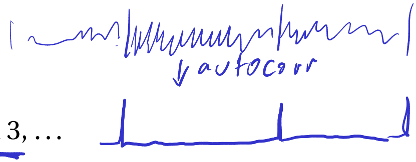
$15 = 2^4 - 1$

8 1's

7 0's

Properties of MLS:

- shift register size m $\Rightarrow$ sequence length $2^m - 1$
- half of the runs: length 1, quarter: length 2, eighth: length 3, ...
- $\approx$ half of bits are 1
- 0 substituted by -1 $\Rightarrow$ crest factor 1 (= minimum)
- correlation property: auto-correlation $\approx$ impulses at intervals of $2^m - 1$



$$(a \star a)[k] = \sum_{t=0}^{2^m-2} a[t]a[t-k] \approx \begin{cases} 2^m - 1 & k = 0 \mod 2^m - 1 \\ 0 & \text{else} \end{cases}$$

So, $a \star a \propto \delta$ (apart from the repetition).

Extract room impulse response h from MLS response $y = h * a$:

$$y \star a = h * (a \star a) = h * \delta = h$$

1 0 0 0 1 0 0 1 1 0 1 0 1 1 1 1
1 0 0 0 1 0 0 1 1 0 1 0 1 1 1

7 equ.
8 7 equ.

Problem: direct convolution of impulse response with input signal computationally costly

Solution: convolution theorem (used on blocks):

$$x * y = F^{-1}\left(F(x)\cdot F(y)\right)$$

n^2 | $n \cdot \log n$, $n \cdot \log n$ | $n \log n$

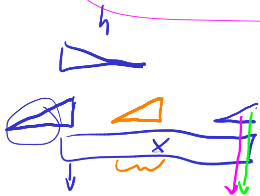
$$\mathrm{FFT}^{-1}\left(\mathrm{FFT}(\underbrace{x[0],\dots,x[n-1]}_{n}) \odot \mathrm{FFT}(\underbrace{\overbrace{h[0],\dots,h[m-1]}^{m},\dots,0}_{\text{length } n})\right)$$

$$= (x[0]\,h[0] + x[n-1]\,h[1] + x[n-2]\,h[2] + \dots, \dots)$$

n

$\odot$ … pointwise multiplication

Problem: result is circular convolution

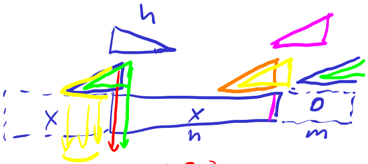


x[0] h[0]   x[n-1] h [1]

Solution: Zero-padding to length $n+m-1$:

$$\mathrm{FFT}^{-1}\Big(\underbrace{\mathrm{FFT}(\overbrace{x[0],\ldots,x[n-1]}^{n},\overbrace{\ldots,0}^{m})}_{\text{length } n+m-1}\odot\underbrace{\mathrm{FFT}(\overbrace{h[0],\ldots,h[m-1]}^{m},\ldots,0)}_{\text{length } n+m-1}\Big)$$

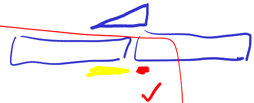
$$= (x[0]h[0], x[1]h[0]+x[0]h[1], \ldots, x[n-1]h[0]+\ldots+x[n-m+1]h[m-1],$$
$$x[n-1]h[1]+\ldots+x[n-m]h[m-1], \ldots, x[n-1]h[m-1]).$$



$x[0]\cdot h[0] + \ldots$

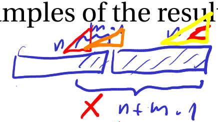
The result has to be overlap-added:

$$
\begin{array}{ll}
x[0]h[0] & \\
x[1]h[0]+x[0]h[1] & \\
\vdots & \\
x[n-1]h[0]+\ldots+x[n-m+1]h[m-1] & \\
x[n-1]h[1]+\ldots+x[n-m]h[m-1] & +x[n]h[0] \\
\vdots & \vdots \\
x[n-1]h[m-1] & +x[n+m-2]h[0]+\ldots+x[n]h[m-2] \\
 & \vdots \\
 & +x[2n-1]h[m-1]\,,
\end{array}
$$



left column: $h*x[0,\ldots,n-1]$, right column: $h*x[n,\ldots,2n-1]$.

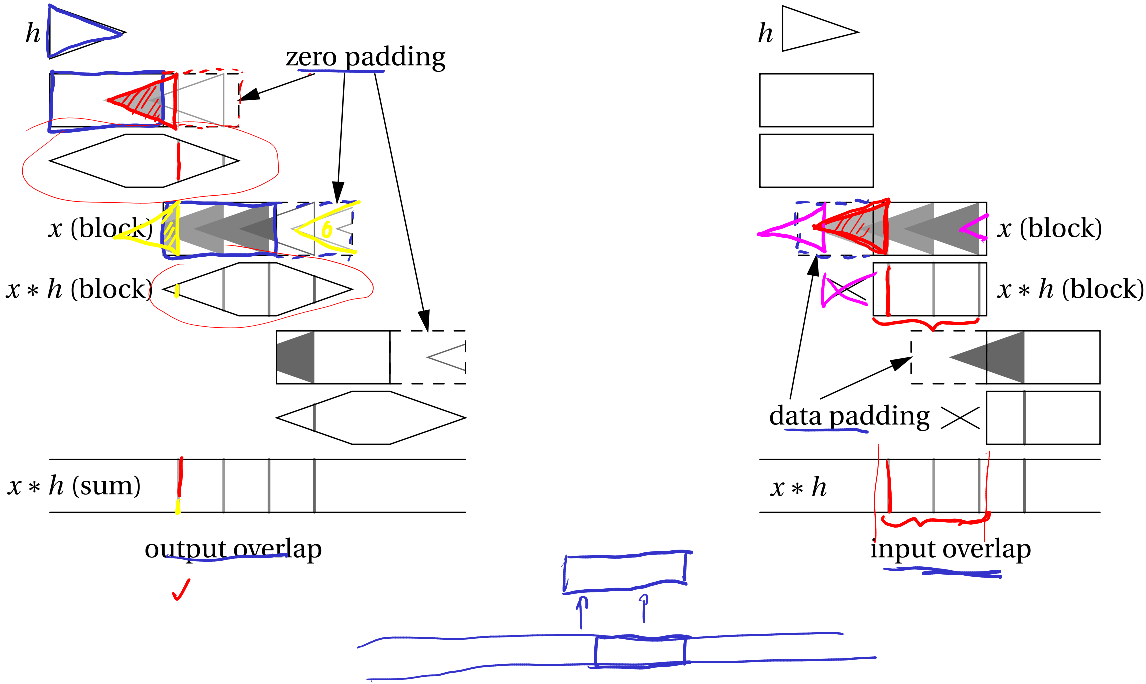
Another possibility: input blocks of size $n+m-1$ overlap, discard $m-1$ samples of the result



$$
\mathrm{FFT}^{-1}\left(\mathrm{FFT}(x[-m+1],\ldots,x[n-1])\odot\mathrm{FFT}(\underbrace{h[0],\ldots,h[m-1],\ldots,0}_{\text{length } n+m-1})\right)
$$

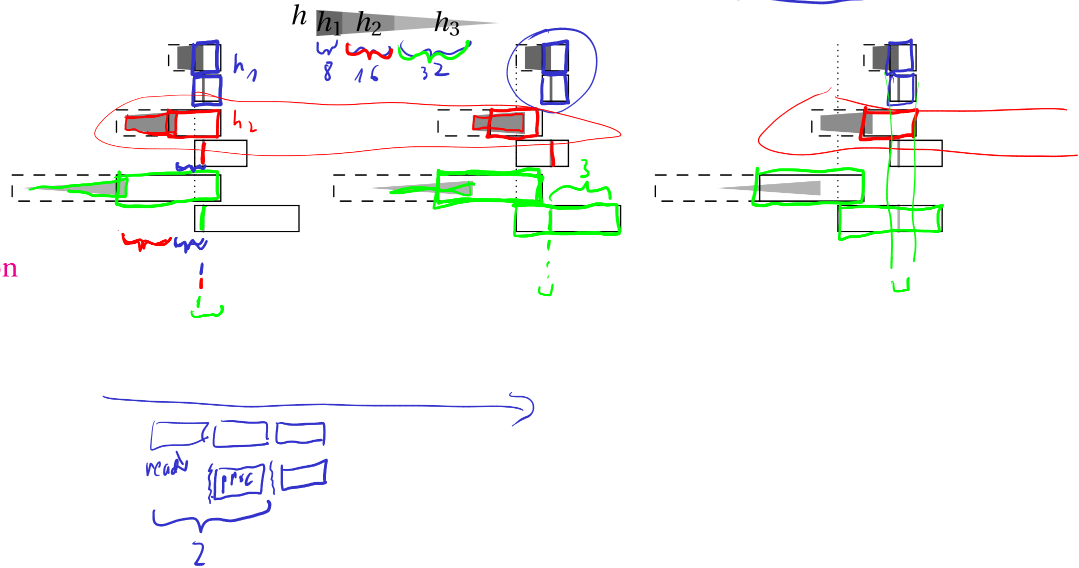
$$
=(\cancel{x[-m+1]h[0]+x[n-1]h[1]+\ldots,\ldots,x[-1]h[0]+\ldots+x[n-1]h[m-1]},
$$

$$
x[0]h[0]+\ldots+x[-m+1]h[m-1],\ldots,x[n-1]h[0]+\ldots+x[n-m]h[m-1])\,.
$$


h
zero padding
x (block)
x ∗ h (block)
x ∗ h (sum)
output overlap
h
x (block)
x ∗ h (block)
data padding
x ∗ h
input overlap

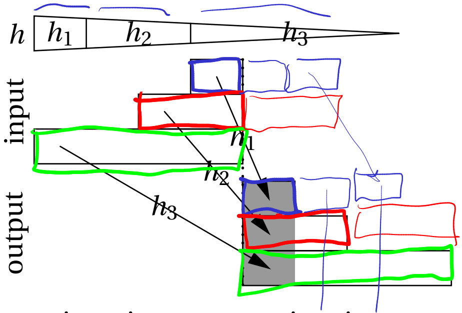
Problem: latency introduced by the block size

Solution: split the impulse response h into blocks $h_1, h_2, h_3, \ldots$ (increasing power-of-two sizes)



animation

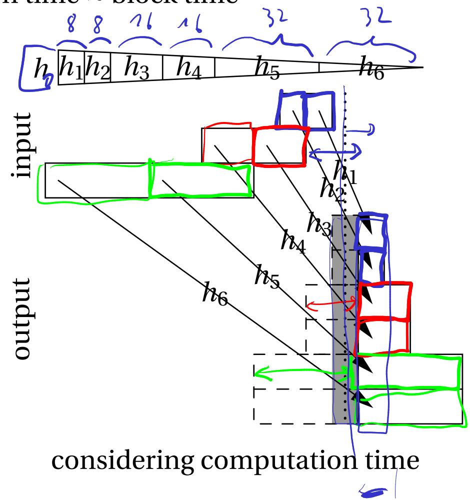
Overlap of input and output of the size of h_1 $\Rightarrow$ introduce some latency



ignoring computation time

animation

Practically: block computation time ≈ block time



considering computation time

animation

Zero-latency: prepend block h_0 ($1\times$ or $2\times$ size of h_1), direct convolution

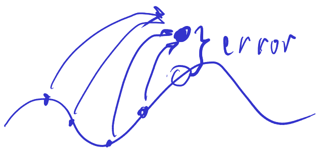
In reality, I/O is blocked anyway, though.

6 Audio Coding

6.1 Lossless Audio Coding

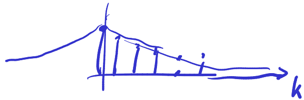
Simplest approach: **silence compression**:
– runs of zero values: runlength-coding
– almost silent parts set to zero (actually lossy)

Better: linear prediction (**linear predictive coding**): LPC
– optimized filter (**Levinson-Durbin recursion**) predicts samples
– encode prediction error



Prediction error has two-sided geometric distribution:

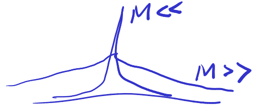
$$p_k = P(x[t] - (p * x)[t] = k) \propto s^{-|k|}$$



Efficiently encoded with **Rice codes**, or Golomb-Rice codes:

– parameter M ($\propto$ variance of the distribution), power of two

– divide k by M $\Rightarrow$ quotient q, remainder r:

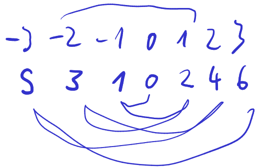


$$k = Mq + r$$

1

– q encoded as unary code (q ones followed by a zero)

– r encoded as $\log_2(M)$ bits



Example ($M = 4$):

q

	k	code	k	code	k	code	k	code
0	0	0 00	4	10 00	8	110 00	12	1110 00
-1	1	0 01	5	10 01	9	110 01	13	1110 01
2	2	0 10	6	10 10	10	110 10	14	1110 10
-2	3	0 11	7	10 11	11	110 11	15	1110 11

3 4

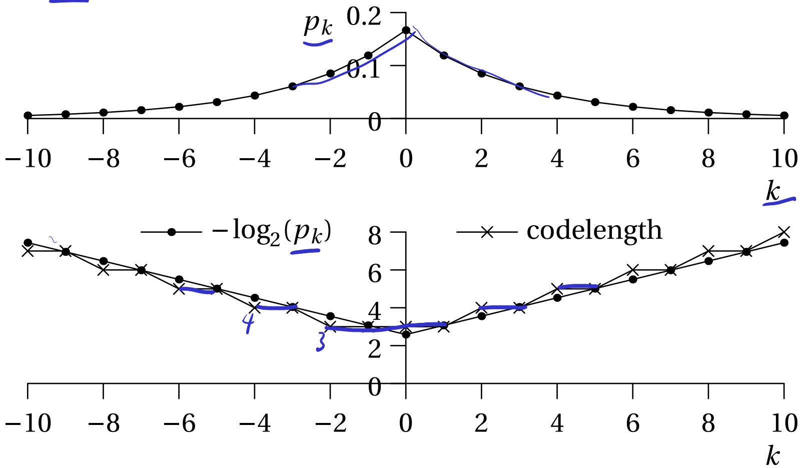
Only suitable for positive k

Signed k: $k \mapsto 2k$ for $k \geq 0$, $k \mapsto 2|k| - 1$ for $k < 0$

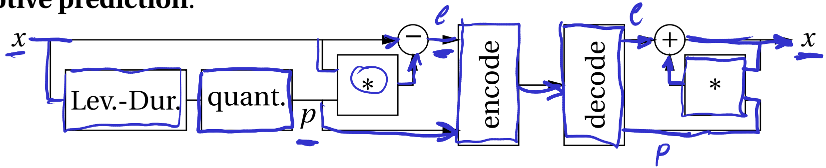
Example: two-sided geometric distribution $p_k = \frac{1}{6} \cdot 1.4^{-|k|}$

Self-information $-\log_2(p_k)$ compared to the codelengths for $M = 4$:

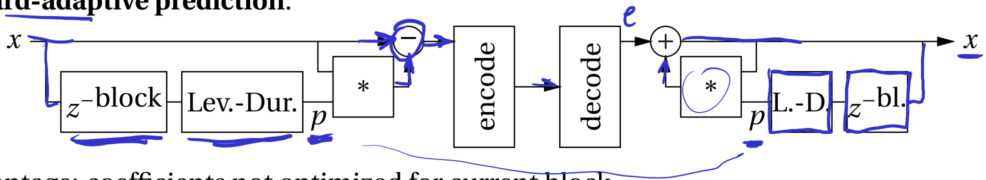
$$\sum_{k=-\infty}^{+\infty} \frac{1}{6}\, 1.4^{-|k|} = 1$$



Forward-adaptive prediction:



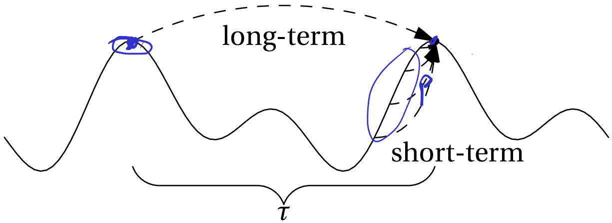
Backward-adaptive prediction:



Disadvantage: coefficients not optimized for current block

Advantages: coefficients not encoded, longer filters possible, non-quantized coefficients

Long-term prediction and short-term prediction:



τ: optimal period (similar to pitch detection)

One to five values around $t - \tau$ for prediction

Short-term and long-term prediction can be combined

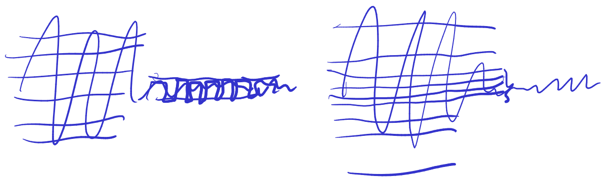
Standards: FLAC (Free Lossless Audio Codec), MPEG-ALS
– many optimization details

6.2 Lossy Audio Coding

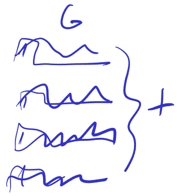
Early simple approaches: μ-law and A-law encoding (logarithmic quantization)

Approaches with linear prediction:

- DPCM (differential pulse code modulation) and ADPCM (adaptive DPCM): only quantized prediction errors encoded
- Pure linear predictive coding: only prediction filter coefficients encoded
- CELP (code excited linear predictor): both encoded



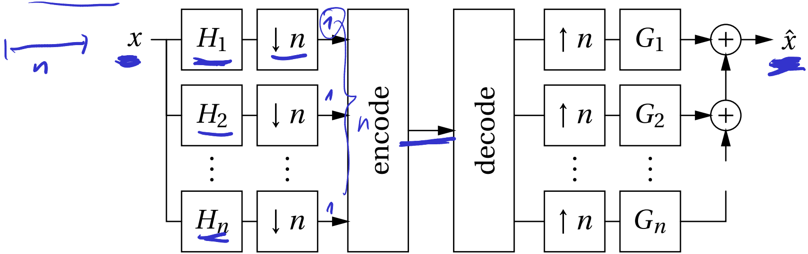
Advanced approach: transform coding (transform of block, quantize and encode coefficients)



Problem: High-frequency artifacts at block borders

Windows and overlapping cannot be used (increase of data size)

Solution 1: filter banks (instead of blocked transform)



H_i . . . bandpass filters with different center frequencies

$\downarrow n$ . . . downsampling by a factor of n

$\uparrow n$ . . . upsampling (insertion of $n-1$ zeros after each element)

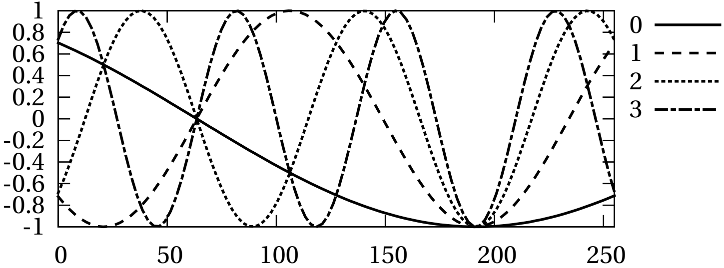
G_i reconstruction filters (H_i and G_i fulfill a "perfect reconstruction" constraint)

Used in MPEG audio level 1–2

Solution 2: **modified discrete cosine transform** (**MDCT**):

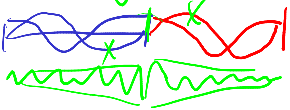
$$X[\boxed{w},\boxed{t}] = \sum_{s=0}^{2n-1} x[nt+s]\cos\left(\frac{\pi}{n}\left(s+\frac{1}{2}+\frac{n}{2}\right)\left(w+\frac{1}{2}\right)\right)$$

n … hop-size $\quad$ $2n$ … block size $\quad$ $w = 0, \ldots, n-1$

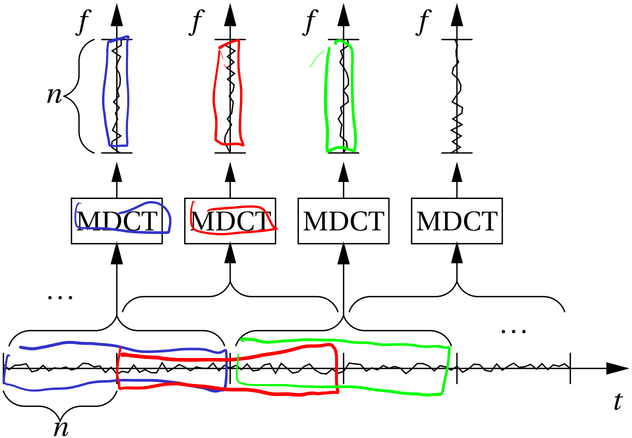


First four basis functions of the MDCT for $n = 128$

cos → DCT



Block of size $2n$ produces only n MDCT coefficients, but 50% overlap of blocks

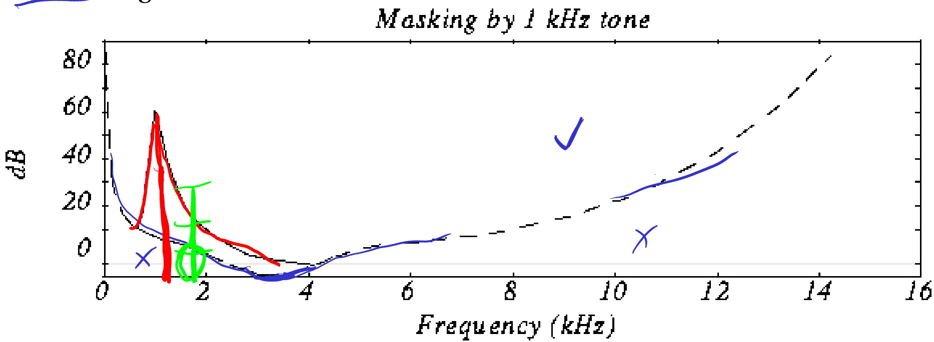


MDCT blocks can be windowed (has to satisfy $w[s]^2 + w[s+n]^2 = 1$)
Used in MPEG audio layer 3 (MP3, in addition to filter banks), MPEG-AAC (advanced audio coding), Vorbis.

Transformed data: quantized and encoded (entropy coders: Huffman, arithmetic coding)

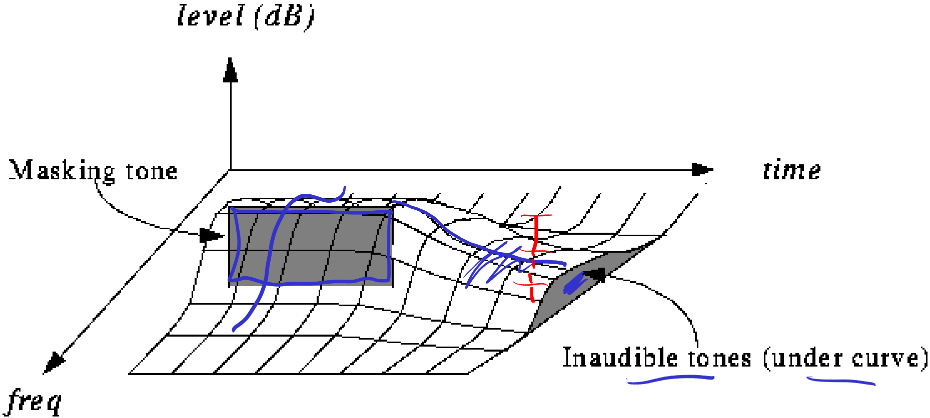
Improvement: adaptively choosing quantization factors on a coefficient basis
⇒ **psychoacoustics**

1. Frequency masking:



⇒ quantize so that quantization is below masking threshold

2. Temporal masking:



Used in all state-of-the-art lossy audio codecs: MP3, AAC, Vorbis

Disadvantages of major audio codecs:

– latency (due to blocked processing ⇒ unusable for interactive audio)
– bad compression performance for very low bit-rate and speech coding
  (predictive techniques still better)
– heavily patent covered techniques

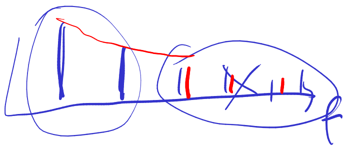
Solution: Opus codec

– frequency-domain techniques for higher bit-rates
– can switch to predictive coding dynamically
– uses small block sizes (less latency) (special techniques to overcome low frequency resolution)

Problem for low bit-rates: high frequencies usually dropped entirely

Solution: **spectral band replication**

- synthesizes higher frequency bands by extrapolating frequency content in lower bands
- harmonic signals supplemented with more harmonic frequencies in higher bands
- low-frequency noise with high-frequency noise
- may be guided by low-bit-rate side information encoded by the encoder
- result: only approximation, but sounds “nice”, improves comprehensibility of speech



The End